

# Φ-MODULES AND COEFFICIENT SPACES

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## INTRODUCTION

This paper is inspired by Kisin's article [Ki1], in which he studies deformations of Galois representations of a local  $p$ -adic field which are defined by finite flat group schemes. The result of Kisin most relevant to our paper is his construction of a kind of resolution of the formal deformation space of the given Galois representation, by constructing a scheme which classifies all finite flat group schemes giving rise to the deformed Galois representation. Our purpose here is to globalize Kisin's construction.

Let  $K$  be a finite extension of  $\mathbb{Q}_p$ , with residue field  $k$ . Let  $K_0$  be the maximal unramified extension of  $\mathbb{Q}_p$  contained in  $K$ . Then  $K_0$  is the fraction field of the ring of Witt vectors  $W = W(k)$ . Let  $\pi$  be a uniformizer of  $K$  and  $E(u) \in W[u]$  the Eisenstein polynomial that  $\pi$  satisfies. Let  $G_K = \text{Gal}(\overline{K}/K)$  be the absolute Galois group of  $K$ . Set  $\mathfrak{S}$  for the ring of formal power series  $W[[u]]$ . Let  $\phi : \mathfrak{S} \rightarrow \mathfrak{S}$  be such that  $\phi|W$  is the Frobenius automorphism and with  $\phi(u) = u^p$ .

Kisin's construction is based on the existence of a fully faithful exact functor from a suitable category of  $\mathfrak{S}$ -modules  $\mathfrak{M}$  equipped with a  $\phi$ -linear endomorphism  $\Phi$  to the category of finite flat (commutative) group schemes of  $p$ -power rank over  $\text{Spec}(\mathcal{O}_K)$ . This in turn was inspired by work of Breuil [Br] who gave a similar but more complicated description of such group schemes. A variant of this functor also works with coefficients: if  $R$  is a  $\mathbb{Z}_p$ -algebra with finitely many elements, then there is a similar functor from a suitable category of  $\mathfrak{S} \otimes_{\mathbb{Z}_p} R$ -modules  $\mathfrak{M}$  with  $\phi$ -linear endomorphism  $\Phi$  to the category of finite flat group schemes of  $p$ -power rank with  $R$ -action over  $\text{Spec } \mathcal{O}_K$ .

Let  $K_\infty/K$  be the extension obtained by adjoining a compatible system of  $p^n$ -power roots of  $\pi$ , and let  $G_{K_\infty} = \text{Gal}(\overline{K}/K_\infty)$  be its absolute Galois group. Let  $\mathcal{O}_{\mathcal{E}}$  be the  $p$ -adic completion of  $\mathfrak{S}[1/u]$ , a complete discrete valuation ring, with uniformizer  $p$  and residue field  $k((u)) = k[[u]][1/u]$ . Then there exists an equivalence of categories between the category of finitely generated  $\mathcal{O}_{\mathcal{E}}$ -modules  $M$  equipped with an isomorphism  $\Phi : \phi^*(M) \rightarrow M$  and the category of continuous representations of  $G_{K_\infty}$  in  $\mathbb{Z}_p$ -modules, and this is compatible with the previous functor via the restriction functor from  $G_K$ -representations to  $G_{K_\infty}$ -representations. Again there is also a variant for representations with values in a finite coefficient  $\mathbb{Z}_p$ -algebra  $R$ .

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Our basic idea in this paper is to formally forget about Galois representations and finite flat group schemes and simply consider the modules themselves, without any finiteness conditions on  $R$ . More precisely, for any  $\mathbb{Z}_p$ -algebra  $R$  set  $R_W = W \otimes_{\mathbb{Z}_p} R$  and extend the endomorphism  $\phi$  of  $W((u)) = W[[u]][1/u]$  to  $R_W((u)) = W((u)) \hat{\otimes}_{\mathbb{Z}_p} R$  by the identity on the second factor. We define the fpqc-stack  $\mathcal{C}_{d,K}$  by giving its values on  $\mathbb{Z}_p$ -algebras  $R$  as the groupoid of pairs  $(\mathfrak{M}, \Phi)$ , where  $\mathfrak{M}$  is a finitely generated  $R_W[[u]]$ -module which is free of rank  $d$  locally fpqc on  $\text{Spec } R$  and where

$$\Phi : \phi^* \mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u]$$

is an isomorphism of  $R_W((u))$ -modules such that  $E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$ .

We also introduce the fpqc-stack  $\mathcal{R}_d$  with values in a  $\mathbb{Z}_p$ -algebra  $R$  the groupoid of pairs  $(M, \Phi)$ , where  $M$  is a finitely generated  $R_W((u))$ -module which is locally fpqc on  $\text{Spec } R$  free of rank  $d$  as  $R_W((u))$ -module, and where  $\Phi : \phi^*(M) \xrightarrow{\sim} M$ .

There is an obvious morphism

$$\theta : \mathcal{C}_{d,K} \rightarrow \mathcal{R}_d$$

sending  $(\mathfrak{M}, \Phi)$  to  $(\mathfrak{M}[1/u], \Phi)$ .

Our main results concern the algebraicity of the previous construction. Let  $\widehat{\mathcal{C}}_{d,K}$  be the  $p$ -adic completion of the stack  $\mathcal{C}_{d,K}$ , i.e. its restriction to  $p$ -nilpotent  $\mathbb{Z}_p$ -algebras. Then

$$\widehat{\mathcal{C}}_{d,K} = \varprojlim_a \mathcal{C}_{d,K} \times_{\text{Spec } \mathbb{Z}_p} \text{Spec } \mathbb{Z}/p^a \mathbb{Z}.$$

Our main result shows that this presents  $\widehat{\mathcal{C}}_{d,K}$  as an inductive 2-limit of Artin stacks of finite type over  $\text{Spec } \mathbb{Z}/p^a \mathbb{Z}$ . Furthermore, the singularities of  $\widehat{\mathcal{C}}_{d,K}$  are modeled by local models.

**Theorem 0.1.** (i) *For each  $a$ , the stack  $\mathcal{C}_{d,K} \times_{\text{Spec } \mathbb{Z}_p} \text{Spec } \mathbb{Z}/p^a \mathbb{Z}$  on  $\mathbb{Z}/p^a \mathbb{Z}$ -algebras is representable by an Artin stack  $\mathcal{C}_{d,K}^a$  of finite type over  $\text{Spec } \mathbb{Z}/p^a \mathbb{Z}$ . The inductive limit  $\varinjlim_a \mathcal{C}_{d,K}^a$  is the formal  $p$ -adic completion  $\widehat{\mathcal{C}}_{d,K}$  of  $\mathcal{C}_{d,K}$ .*

(ii) *There is a “local model” diagram*

$$\begin{array}{ccc} & \widetilde{\widehat{\mathcal{C}}_{d,K}} & \\ \pi \swarrow & & \searrow \varphi \\ \widehat{\mathcal{C}}_{d,K} & & \widehat{M}_{d,K} , \end{array}$$

in which  $\pi$  is a principal homogeneous space under the positive loop group  $L^+ G$  of  $G = \text{Res}_{W/\mathbb{Z}_p}(GL_d)$  completed along its special fiber, and in which  $\varphi$  is formally smooth. Here  $M_{d,K}$  is the projective  $\mathbb{Z}_p$ -scheme parametrizing all  $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ -submodules of  $(\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_S)^d$  which are locally direct summands as  $\mathcal{O}_S$ -modules, and  $\widehat{M}_{d,K}$  denotes its completion along the special fiber.

(iii) For each  $a$ , set  $\mathcal{R}_d^a = \mathcal{R}_d \times_{\text{Spec } \mathbb{Z}_p} \text{Spec } \mathbb{Z}/p^a \mathbb{Z}$ . The morphism  $\theta^a : \mathcal{C}_{d,K}^a \rightarrow \mathcal{R}_d^a$  given by reducing  $\theta$  modulo  $p^a \mathbb{Z}$  is representable and proper; hence  $\hat{\theta} : \widehat{\mathcal{C}}_{d,K} \rightarrow \widehat{\mathcal{R}}_d$  is an inductive limit of representable and proper morphisms.

The fibers of  $\theta$  over a finite field  $\mathbb{F}$  are interesting projective subvarieties of the affine Grassmannian of the group  $G = \text{Res}_{W/\mathbb{Z}_p}(GL_d)$ , which we call *Kisin varieties*. More precisely, we define variants of  $\mathcal{C}_{d,K}$  and  $M_{d,K}$  depending on a co-character  $\mu$  of  $G$  and define Kisin varieties associated to  $(G, A, \mu)$ , where  $A \in G(\mathbb{F}((u)))$  defines the given  $\mathbb{F}$ -valued point in  $\mathcal{R}_d$ . They are the analogues, for the kind of Frobenius involved here, of the affine Deligne-Lusztig varieties appearing in the isocrystal context, cf., eg. [GHKR]. The study of these varieties was begun by Kisin in [Ki1], in the case  $d = 2$  and  $W = \mathbb{Z}_p$ . In a companion paper to ours, E. Hellmann extends Kisin's results (again for  $d = 2$  and  $W = \mathbb{Z}_p$ ). In Hellmann's paper, one of the main tools is the Bruhat-Tits building of  $GL_2$ . We show here how the Bruhat-Tits building can be used in general to gain a qualitative overview of Kisin varieties.

The other extreme to the fiber over  $\mathbb{F}$  of  $\widehat{\mathcal{C}}_{d,K}$  is its fiber over  $\mathbb{Q}_p$ . Here we construct a kind of period map of stacks over the category of adic formal schemes locally of finite type over  $\text{Spf}(\mathbb{Z}_p)$ ,

$$\Pi(\mathcal{X}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \rightarrow \mathfrak{D}_{d,K}(\mathcal{X}^{\text{rig}}),$$

where  $\mathfrak{D}_{d,K}$  is the stack over the category of rigid-analytic spaces over  $\mathbb{Q}_p$  parametrizing filtered  $\Phi$ -modules (both the filtration and  $\Phi$  vary!). To determine the image of the period map seems one of the major challenges in the theory. We conjecture that the image should be given somewhat analogously to Hartl's *admissible set* in [Ha].

As is apparent from the above, the theory developed here bears many similarities to the theory of period spaces for  $p$ -divisible groups in [RZ], but there are also substantial differences. The stack  $\widehat{\mathcal{C}}_{d,K}$  is analogous to one of the period spaces of  $p$ -divisible groups in [RZ], but unlike these it is adic over  $\text{Spec}(\mathbb{Z}_p)$  ( $p$  generates an ideal of definition); the local model diagram looks formally just like the corresponding one in [RZ]; Kisin varieties are the analogues of affine Deligne-Lusztig varieties, and the stack  $\mathfrak{D}_{d,K}$  plays a role similar to the Grassmannian containing the period space of [RZ]. In [RZ], the base scheme of the  $p$ -divisible groups is variable; here the base scheme  $\text{Spec}(\mathcal{O}_K)$  is constant, but the coefficients are variable.

We now explain the lay-out of the paper. In section 1 we explain by analogy on the classical theory of unit root crystals the spaces/stacks we encounter. In section 2 we prove our main technical result, which states that the stack  $\mathcal{C}_d$ , which associates to  $\mathbb{Z}_p$ -algebras  $R$  the groupoid of locally free  $R_W[[u]]$ -modules of rank  $d$  with  $\Phi$ -module structure, can be presented as an inductive limit of Artin stacks of finite type over  $\mathbb{Z}_p$ . In section 3 we fix the local field as above and prove the main theorem stated above. In section 4 we indicate the relation to the deformation spaces of Galois representations which is at the origin of Kisin's

theory. In section 5 we construct and discuss the period morphism. In the final section 6 we discuss Kisin varieties and their analysis through Bruhat-Tits buildings.

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## 1. MOTIVATION: UNIT CRYSTALS AND GALOIS REPRESENTATIONS

**1.a. Unit root crystals.** Let  $k$  be a finite field of characteristic  $p > 0$ ; for simplicity, we will assume that  $k = \mathbb{F}_p$ . Let  $X$  a variety over  $k$ . Denote by  $\phi : X \rightarrow X$  the Frobenius  $\phi(a) = a^p$  for  $a \in \mathcal{O}_X$ . Suppose that  $S$  is a  $k$ -scheme, and let  $\phi_S = \phi \times id_S : X \times S \rightarrow X \times S$ . Consider pairs  $(\mathcal{M}, F)$  consisting of a locally free  $\mathcal{O}_{X \times S}$ -coherent sheaf  $\mathcal{M}$  of rank  $d$  on  $X \times S$  and an isomorphism

$$F : \phi_S^* \mathcal{M} \xrightarrow{\sim} \mathcal{M} .$$

As  $S$  varies, these pairs form an fpqc stack  $FM_X^{d,et}$  over  $\text{Spec } k$ . In fact,  $FM_X^{d,et}$  is an Artin stack locally of finite type over  $k$ . Indeed, let  $Fib_{X/k}^d$  be the Artin stack locally of finite type over  $k$ , whose values in a  $k$ -scheme  $S$  is the groupoid of locally free  $\mathcal{O}_{X \times S}$ -modules of rank  $d$ , and let  $\widetilde{Fib}_{X/k}^d$  be the  $GL_d$ -torsor over  $Fib_{X/k}^d$ , consisting of a locally free  $\mathcal{O}_{X \times S}$ -modules  $\mathcal{M}$  of rank  $d$  and a basis  $\iota : \mathcal{M} \xrightarrow{\sim} \mathcal{O}_{X \times S}^d$ . Then there is an action of  $GL_d$  on the product  $\widetilde{Fib}_{X/k}^d \times GL_d$  via

$$g : (\mathcal{M}, \iota, A) \mapsto (\mathcal{M}, g^{-1} \cdot \iota, g^{-1} \cdot A \cdot \phi(g)) .$$

This presents  $FM_X^{d,et}$  as a quotient of  $\widetilde{Fib}_{X/k}^d \times GL_d$  by  $GL_d$ , and hence  $FM_X^{d,et}$  is an Artin stack locally of finite type, as claimed.

Suppose  $S = \text{Spec}(\Lambda)$  with  $|\Lambda| < \infty$ . Then by Katz [Ka], 4.1, cf. also [E-K], there is a bijective correspondence between pairs  $(\mathcal{M}, F)$  over  $\text{Spec}(\Lambda)$  and étale sheaves of  $\Lambda$ -modules on  $X$ . In the case  $\Lambda = \mathbb{F}_p$ , this correspondence is obtained via push-out from the injection  $GL_d(\mathbb{F}_p) \rightarrow GL_d$  which induces an equivalence of categories between the category of  $GL_d(\mathbb{F}_p)$ -torsors on  $X$  (for the étale topology) and the category of  $GL_d$ -torsors  $P$  on  $X$  with an isomorphism  $\phi^*(P) \xrightarrow{\sim} P$ .

We want to think of  $FM_X^{d,et}$  as a “coefficient space” for  $p$ -torsion representations of  $\pi_1(X, \bar{\eta})$ . However, it seems that global questions on these spaces (i.e., when  $S$  is not a local Artin ring) have not been studied much in the literature. For instance, are there non-constant morphisms of projective  $k$ -schemes into  $FM_X^{d,et}$ ? What is the dimension of  $FM_X^{d,et}$ ? Etc. The only result we are aware of is Laszlo’s construction [La] of a projective curve  $X$  of genus 2 over the field with 2 elements, a projective curve  $S$  over a finite extension  $k'$  of  $\mathbb{F}_2$  and a locally free coherent  $\mathcal{O}_{X \times S}$ -module  $\mathcal{M}$  of arbitrary rank with an isomorphism  $F : (\phi_S^2)^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ .

**1.b. Variants.** We mention here some variants of the above theory. Let  $G$  be a reductive group over  $\mathbb{F}_p$ . Then we can consider the fpqc stack  $FM_X^{G,et}$  of pairs  $(P, F)$ , where  $P$  is a  $G$ -torsor on  $X \times S$  and where  $F : \phi_S^*(P) \xrightarrow{\sim} P$ . For  $G = GL_d$ , we recover the stack considered above.

We may also consider “meromorphic Frobenius structures”, as follows. Assuming  $X$  to be irreducible, with generic point  $\eta(X)$ , consider the fpqc stack  $FM_X^d$  of pairs  $(\mathcal{M}, F)$  with  $\mathcal{M}$  a locally free  $\mathcal{O}_{X \times S}$ -coherent sheaf of rank  $d$  on  $X \times S$  and

$$F : \phi_S^*\mathcal{M} \rightarrow \mathcal{M}$$

a homomorphism such that  $F|_{\eta(X) \times S}$  is an isomorphism.

One may also control the degeneracy of the meromorphic Frobenius structure. For instance, let  $X$  be a curve. Then we may consider triples  $(\mathcal{M}, F, x)$  with  $(\mathcal{M}, F)$  in  $FM_X^d$  and  $x : S \rightarrow X$  such that  $Coker(F)$  is supported on the graph  $\Gamma_x \subset X \times S$  and is annihilated by the power of the ideal sheaf  $I_{\Gamma_x}^e$  for some fixed  $e \geq 1$ . Denoting the corresponding stack by  $FM_{X,e}^d$ , there is a morphism (the “pole morphism”),

$$p : FM_{X,e}^d \rightarrow X$$

Similarly we can obtain a construction that resembles shtuka, but the Frobenius is “on the other factor”. Namely, assume that  $X$  is a curve as before. Consider  $(\mathcal{M}, \mathcal{M}', F, F', x, y)$  with  $\mathcal{M}, \mathcal{M}'$  locally free  $\mathcal{O}_{X \times S}$ -coherent sheaves of rank  $d$  on  $X \times S$  and homomorphisms

$$\phi_S^*\mathcal{M} \xrightarrow{F} \mathcal{M}' \xleftarrow{F'} \mathcal{M}$$

such that  $Coker(F)$ , resp.  $Coker(F')$  is supported on the graph of  $x : S \rightarrow X$ , resp.  $y : S \rightarrow X$ . Again we can ask that  $Coker(F)$ , resp.  $Coker(F')$ , satisfy some additional property.

Another variant is obtained by replacing the variety  $X$  by the spectrum of the completed local ring at a closed point, or by its fraction field.

All these “spaces”/stacks seem interesting geometric objects.

## 2. SPACES OF KISIN-BREUIL MODULES

Fix a finite field  $k$  of characteristic  $p$  and denote by  $\phi(a) = a^p$  the Frobenius automorphism of  $k$ . We will denote by  $W = W(k)$  the ring of Witt vectors of  $k$  and by  $\phi : W \rightarrow W$  the unique lifting of Frobenius.

Let  $R$  be a commutative  $\mathbb{Z}_p$ -algebra and set  $R_W = W \otimes_{\mathbb{Z}} R$ . We extend  $\phi$  in a  $R$ -linear way to  $R_W$  and denote this extension also by  $\phi$ . We also denote by  $\phi$  the endomorphism  $\phi$  of  $R_W((u)) = W((u)) \hat{\otimes}_{\mathbb{Z}} R$  given by

$$\phi\left(\sum_i a_i u^i\right) = \sum_i \phi(a_i) u^{pi}.$$

2.a. We define now various stacks of modules with Frobenius structure.

Let us consider the stack  $\mathcal{C}_d$  such that  $\mathcal{C}_d(R)$  is the groupoid of  $R_W[[u]]$ - $\Phi$ -modules  $(\mathfrak{M}, \Phi)$ : These are by definition pairs of a  $R_W[[u]]$ -module  $\mathfrak{M}$  which is locally on  $R$  (for the fpqc topology)  $R_W[[u]]$ -free of rank  $d$  and a  $R_W((u))$ -module isomorphism

$$\Phi : \phi^*\mathfrak{M}[1/u] = R_W((u)) \otimes_{\phi, R_W[[u]]} \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}[1/u] = R_W((u)) \otimes_{R_W[[u]]} \mathfrak{M} .$$

It is easy to see that  $\mathcal{C}_d$  is a stack for the fpqc topology.

Next, consider the stack  $\mathcal{R}_d$  which is such that  $\mathcal{R}_d(R)$  is the groupoid of pairs  $(M, \Phi)$  of  $R_W((u))$ -modules  $M$  which are fpqc locally on  $R$  free of rank  $d$ , together with a  $R_W((u))$ -linear isomorphism

$$\Phi : \phi^*M := R_W((u)) \otimes_{\phi, R_W((u))} M \rightarrow M .$$

Again it is easy to see that  $\mathcal{R}_d$  gives a stack for the fpqc topology. Write

$$\theta : \mathcal{C}_d \rightarrow \mathcal{R}_d ; \quad (\mathfrak{M}, \Phi) \mapsto (\mathfrak{M}[1/u], \Phi)$$

for the forgetful morphism.

Fix an integer  $m \geq 0$ . Let us consider the stack  $\mathcal{C}_{m,d}$  such that  $\mathcal{C}_{m,d}(R)$  is the groupoid of  $R_W[[u]]$ - $\Phi$ -modules  $(\mathfrak{M}, \Phi)$  as above that satisfy the additional hypothesis

$$(2.1) \quad u^m \mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset u^{-m} \mathfrak{M} .$$

Once again,  $\mathcal{C}_{m,d}$  gives a stack for the fpqc topology. The natural morphism  $\mathcal{C}_{m,d} \rightarrow \mathcal{C}_d$  is a representable closed immersion.

If  $d$  is fixed we will often write  $\mathcal{C}$ ,  $\mathcal{R}$ ,  $\mathcal{C}_m$  instead of  $\mathcal{C}_d$ ,  $\mathcal{R}_d$  and  $\mathcal{C}_{m,d}$ .

2.b. For simplicity, we will set  $G = \text{Res}_{W/\mathbb{Z}_p} GL_d$ . Set

$$\begin{aligned} LG(R) &:= GL_d(R_W((u))) , \\ L^+G(R) &:= GL_d(R_W[[u]]) , \\ LG^{\leq m}(R) &:= \{A \in GL_d(R_W((u))) \mid A, A^{-1} \in u^{-m} M_d(R_W[[u]])\} . \end{aligned}$$

Hence  $L^+G = LG^{\leq 0}$ . Note that the functor

$$R \mapsto LG^{\leq m}(R)$$

is represented by a scheme  $LG^{\leq m}$  (which is infinite dimensional). Let  $(\mathfrak{M}, \Phi) \in \mathcal{C}_m(R)$  such that  $\mathfrak{M}$  is a free  $R_W[[u]]$ -module. By picking a  $R_W[[u]]$ -basis of  $\mathfrak{M}$ , we can write  $\Phi$  as multiplication by  $A \in LG^{\leq m}(R)$ . Changing the basis by  $g \in GL_d(R_W[[u]])$  amounts to changing  $A$  to  $g^{-1} \cdot A \cdot \phi(g)$ . Therefore, we can write

$$(2.2) \quad \mathcal{C}_{m,d} = [LG^{\leq m} /_{\phi} L^+G]$$

where the quotient  $/_{\phi}$  is via the right action of  $L^+G(R) = GL_d(R_W[[u]])$  by  $\phi$ -conjugation by  $A \star g = g^{-1} \cdot A \cdot \phi(g)$ .

Similarly, we can write

$$(2.3) \quad \mathcal{C}_d = [LG/\phi L^+G], \quad \mathcal{R}_d = [LG/\phi LG].$$

In fact, we can consider the fpqc stack  $\tilde{\mathcal{C}}_d$  defined as follows:  $\tilde{\mathcal{C}}_d(R)$  is the groupoid of pairs  $((\mathfrak{M}, \Phi), \alpha)$  of  $R_W[[u]]$ - $\Phi$ -modules  $(\mathfrak{M}, \Phi)$  together with an  $R_W[[u]]$ -module isomorphism

$$\alpha : R_W[[u]]^d \xrightarrow{\sim} \mathfrak{M}.$$

The stack  $\tilde{\mathcal{C}}_d$  is represented by the ind-scheme  $LG$  and the forgetful morphism

$$\pi : \tilde{\mathcal{C}}_d \rightarrow \mathcal{C}_d$$

is a  $L^+G$ -torsor.

2.b.1. Denote by  $\mathcal{F}_G = LG/L^+G$  the affine Grassmannian of  $R_W[[u]]$ -“lattices” in  $R_W((u))^d$ . (Here by  $R_W[[u]]$ -lattice we mean a locally on  $R$  free  $R_W[[u]]$ -submodule  $L$  of  $R_W((u))^d$  such that  $L \otimes_{R[[u]]} R((u)) = R((u))^d$ .) The fpqc quotient  $\mathcal{F}_G = LG/L^+G$  is represented by an ind-scheme which is ind-projective over  $\mathbb{Z}_p$ . For  $m \geq 0$ , let  $\mathcal{F}_G^{\leq m}$  be the projective subscheme of  $\mathcal{F}_G$  parametrizing  $R_W[[u]]$ -lattices  $L$

$$u^m R_W[[u]]^d \subset L \subset u^{-m} R_W[[u]]^d.$$

(This is a finite union of Schubert varieties in the affine Grassmannian.) Set  $U_0(R) = L^+G(R) = GL_d(R_W[[u]])$  and define for  $n \geq 1$  the principal congruence subgroup  $U_n$  of level  $n$  by  $U_n(R) = I + u^n \cdot M_d(R_W[[u]])$ . The subgroup scheme  $U_n$  is normal in  $L^+G$  and the quotient  $L^+G/U_n$  is represented by the smooth finite type group scheme  $\mathcal{G}_n$  given by the Weil restriction of  $GL_d$  from  $W[[u]]/(u^n)$  to  $\mathbb{Z}_p$  (so that  $\mathcal{G}_n(R) = GL_d(R_W[[u]]/(u^n))$ ). Note that under the action of  $L^+G$  on  $\mathcal{F}_G^{\leq m}$  the subgroup  $U_{2m}$  acts trivially, and hence the action factors through  $\mathcal{G}_{2m}$ .

**Theorem 2.1.** a) For  $m \geq 1$ ,  $\mathcal{C}_m = [LG^{\leq m}/\phi L^+G]$  is an Artin stack of finite type over  $\mathbb{Z}_p$ . We can write  $\mathcal{C}$  as a direct 2-limit

$$\mathcal{C} = \varinjlim_m \mathcal{C}_m$$

and so  $\mathcal{C}$  is an “ind-Artin stack of ind-finite type over  $\mathbb{Z}_p$ ”.

b) There is a formally smooth morphism

$$q : \mathcal{C} \rightarrow [L^+G \setminus \mathcal{F}_G] = [L^+G \setminus LG/L^+G].$$

In fact,  $q$  is given as the limit of formally smooth morphisms

$$q_m : \mathcal{C}_m \rightarrow [L^+G \setminus \mathcal{F}_G^{\leq m}] = [L^+G \setminus LG^{\leq m}/L^+G].$$

The composition of  $q_m$  with the natural morphism  $[L^+G \setminus \mathcal{F}_G^{\leq m}] \rightarrow [\mathcal{G}_{2m} \setminus \mathcal{F}_G^{\leq m}]$  is a smooth morphism of Artin stacks of finite type,

$$\bar{q}_m : \mathcal{C}_m \rightarrow [\mathcal{G}_{2m} \setminus \mathcal{F}_G^{\leq m}].$$

The relative dimension of  $\bar{q}_m$  is equal to  $2md^2$ .

*Proof.* The group  $LG(R)$  is a topological group with topology described by the neighborhoods  $U_n$  of the identity  $I = I_d$ . We have

$$LG(R) = \bigcup_{m \geq 0} LG^{\leq m}(R).$$

Suppose now that  $A$  is in  $LG^{\leq m}(R)$ . For any integer  $n \geq 0$  and for all  $A'$  in the neighborhood (coset)

$$\{A' \mid A' \cdot A^{-1} \in U_n(R)\} = U_n(R) A$$

of  $A$ , we have  $A' \in LG^{\leq m}(R)$  also.

**Proposition 2.2.** *Suppose  $n > 2m/(p - 1)$ .*

- 1) *For each  $g \in U_n(R)$ ,  $A \in LG^{\leq m}(R)$ , we can write  $g^{-1} \cdot A \cdot \phi(g) = H^{-1} \cdot A$  with a unique  $H = H(g, A) \in U_n(R)$ .*
- 2) *Conversely, for each  $A \in LG^{\leq m}(R)$  and  $h \in U_n(R)$ , there is a unique  $g \in U_n(R)$  such that  $A \star g = g^{-1} \cdot A \cdot \phi(g) = h^{-1} \cdot A$ .<sup>1</sup>*

*Proof.* Let us first prove (1). Write  $g^{-1} = I + u^n X$  with  $X \in M_d(R_W[[u]])$ . Then  $\phi(g) = I + u^{pn} Y$ , with  $Y \in M_d(R_W[[u]])$ . Now

$$\begin{aligned} g^{-1} \cdot A \cdot \phi(g) \cdot A^{-1} &= (I + u^n X) \cdot A \cdot (I + u^{pn} Y) \cdot A^{-1} = \\ &= (I + u^n X) \cdot (I + u^{pn} A Y A^{-1}). \end{aligned}$$

Observe that  $AYA^{-1} \in u^{-2m} M_d(R_W[[u]])$  and  $pn - 2m > n$ . Hence, for

$$H^{-1} = (I + u^n X) \cdot (I + u^{pn} A Y A^{-1})$$

we obtain  $g \cdot A \cdot \phi(g)^{-1} = H^{-1} \cdot A$ . The element  $H$  is uniquely determined from  $g$  and  $A$  by  $g \cdot A \cdot \phi(g)^{-1} = H^{-1} \cdot A$ .

The statement (2) is little trickier. First we show that if such a  $g$  exists it is uniquely determined by  $h$  and  $A$ . It is enough to assume  $g \cdot A \cdot \phi(g)^{-1} = A$  with  $g \in U_n(R)$  and  $A \in LG^{\leq m}(R)$  and show  $g = 1$ . Write  $g = I + u^n X$ ,  $\phi(g) = I + u^{pn} \phi(X)$ . We have

$$(I + u^n X) \cdot A = A \cdot (I + u^{pn} \phi(X))$$

which gives

$$u^n X \cdot A = u^{pn} A \cdot \phi(X),$$

i.e.,

$$X_0 + X_1 u + X_2 u^2 + \dots = u^{(p-1)n} A \cdot (X_0 + X_1 u^p + X_2 u^{2p} + \dots) A^{-1}$$

Note that  $A \cdot X_i \cdot A^{-1} \in u^{-2m} M_d(R[[u]])$ . Since  $(p-1)n - 2m > 0$ , we obtain  $X_0 = 0$  which implies  $g \in U_{n+1}(R)$ . An induction finishes the proof of uniqueness.

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<sup>1</sup>The fact that two elements  $A$  and  $A'$  in  $GL_d(\bar{\mathbb{F}}_p((u)))$  which are  $u$ -adically close, are  $\phi$ -conjugate is also used by Caruso in [Ca]. The analogous fact for classical Dieudonné modules is also true.

Now we will show that such a  $g$  exists. Let  $A' = h^{-1}A$ . Set  $A_0 = A$ ,  $h_0 = h$  and define  $h_i$  and  $A_i$  inductively by

$$(2.4) \quad A_i = h_{i-1}^{-1} \cdot A_{i-1} \cdot \phi(h_{i-1}), \quad A' = h_i^{-1} \cdot A_i.$$

Set  $\kappa(i) := p^i n - 2(1 + p + \dots + p^{i-1})m$  with  $\kappa(0) = n$ . Note that under our assumption  $n > 2m/(p-1)$ , the function  $\kappa(i)$  strictly increases with  $i \geq 0$ .

The existence of  $g$  will now follow from

**Lemma 2.3.** 1) We have  $h_i \in U_{\kappa(i)}(R)$  and so  $\lim_{i \rightarrow \infty} h_i = I$ .

2) Let  $g_i = \prod_{j=0}^i h_j$ . Then the limit  $g = \lim_{i \rightarrow \infty} g_i$  exists and belongs to  $U_n(R)$ .

*Proof.* We will prove (1) by induction. It is true by our hypothesis when  $i = 0$ . The equalities (2.4) imply

$$h_i = A' \cdot \phi(h_{i-1}) \cdot A'^{-1}.$$

By the induction hypothesis  $h_{i-1} \in U_{\kappa(i-1)}(R)$  so

$$\phi(h_{i-1}) = I + \phi(u^{\kappa(i-1)}X) = I + u^{p \cdot \kappa(i-1)}\phi(X)$$

with  $X \in M_d(R_W[[u]])$ . Since  $A' \in LG^{\leq m}(R)$ , we have

$$A' \cdot \phi(X) \cdot A'^{-1} \in u^{-2m}M_d(R_W[[u]]),$$

and so

$$h_i = A' \cdot \phi(h_{i-1}) \cdot A'^{-1} = I + u^{p \cdot \kappa(i-1)-2m}Y$$

with  $Y \in M_d(R_W[[u]])$ . Since  $\kappa(i) = p \cdot \kappa(i-1) - 2m$  this completes the proof of (1). Part (2) now follows immediately since from part (1)

$$g_i = \prod_{j=0}^i h_j = (I + u^{\kappa(0)}X_0) \cdot (I + u^{\kappa(1)}X_1) \cdot \dots \cdot (I + u^{\kappa(i)}X_i)$$

with  $X_j \in M_d(R_W[[u]])$  and  $i \mapsto \kappa(i)$  is strictly increasing.  $\square$

Now  $h_i^{-1}A_i = g_i^{-1} \cdot A \cdot \phi(g_{i-1})$ , hence passing to the limit, we obtain  $g^{-1} \cdot A \cdot \phi(g) = A' = h^{-1} \cdot A$  as desired.  $\square$

**Remark 2.4.** Let  $M$  be a  $R((u))$ -module and let  $R \rightarrow R'$  be a flat extension such that  $M' = M \hat{\otimes}_R R' \simeq R'((u))^d$ . The module  $M$  has a natural topology as a Tate  $R$ -module (see [Dr]). The  $R$ -lattices of  $M$  (i.e  $R$ -modules  $L$  which are open and such that  $L/U$  is finitely generated for every open submodule  $U \subset L$ ) give a basis of open neighborhoods of 0. Multiplication by  $u$  on  $M$  is topologically nilpotent; i.e given any two  $R$ -lattices  $L, L'$ , there is  $N \geq 0$  such that  $u^N \cdot L \subset L'$ .

Then  $\mathcal{G}_M := \text{Aut}_{R((u))}(M)$  has a natural structure of a topological group. To obtain a basis of neighborhoods of the identity  $I$  we take a lattice  $L$  and for  $n \gg 0$  we consider

$$U_n(L) = \{g \in \mathcal{G}_M \mid g(L) \subset L, \quad g|_L \equiv I \pmod{u^n L}\}.$$

We can then show:

Given two isomorphisms  $A, A' : \phi^*M \xrightarrow{\sim} M$ , there exists an open neighborhood  $U$  of the identity in the topological group  $\mathcal{G}_M$  such that if  $A' \cdot A^{-1} \in U$  then  $A, A'$  are  $\phi$ -conjugate by a uniquely determined element of  $\mathcal{G}_M$ .

The argument is similar as above: Suppose that  $A : \phi^*M \xrightarrow{\sim} M$  is an  $R((u))$ -isomorphism and for  $h \in \mathcal{G}_M$  define  $h_0 = h$  and inductively

$$h_i = A \cdot \phi^*(h_{i-1}) \cdot A^{-1}.$$

The result follows from the statement: There is an open neighborhood  $U$  of  $I$  such that for  $h \in U$ , we have  $\lim_{i \rightarrow \infty} h_i = I$  and the limit  $\tilde{h} = \lim_{i \rightarrow \infty} \prod_{0 \leq j \leq i} h_j$  exists. Indeed, the arguments above show that this is true when  $M$  is a free  $R((u))$ -module. In general, let  $R \rightarrow R'$  be a flat homomorphism such that  $M' = M \hat{\otimes}_R R' \simeq R'((u))^d$ . Consider  $h'_i = h_i \otimes 1 \in \mathcal{G}_{M'}$  and let  $L$  be a lattice in  $M$ . Then  $L' = L \hat{\otimes}_R R'$  is a lattice in  $M'$ . By the above, there is  $n$  such that when  $h \in U_n(L)$  (and hence  $h' = h \otimes 1 \in U_n(L')$ ) we have

$$(h_i(x) - x) \otimes 1 = h'_i(x') - x' \in u^{\kappa(i)} L'$$

for a strictly increasing sequence  $\kappa(i)$ . Since  $u^{\kappa(i)} L' \cap M = u^{\kappa(i)} L$  this shows the result.

We now continue with the proof of Theorem 2.1 (a). Recall

$$\mathcal{C}_m = [LG^{\leq m}/\phi L^+ G]$$

Let  $n > 2m/(p-1)$ . Recall the normal subgroup  $U_n$  of  $L^+ G$  and its smooth finite type quotient  $\mathcal{G}_n$ . Consider the quotient stack  $[LG^{\leq m}/\phi U_n]$ . Proposition 2.2 implies that  $[LG^{\leq m}/\phi U_n]$  coincides with the quotient  $X_{n,d}^{\leq m} := [LG^{\leq m}/U_n]$  by the free translation action of  $U_n$  on  $LG^{\leq m}$ . The quotient  $X_{n,d}^{\leq m}$  is represented by a scheme of finite type over  $\mathbb{Z}_p$ . This can be seen as follows.

Recall that the quotient  $X_{0,d}^{(m)} = [LG^{\leq m}/L^+ G]$  is represented by the closed subscheme  $\mathcal{F}_G^{\leq m}$  of the affine Grassmannian  $\mathcal{F}_G = LG/L^+ G$  that parametrizes lattices  $L$  such that  $u^m L_0 \subset L \subset u^{-m} L_0$ , where  $L_0 = R_W[[u]]^d$ . The natural map

$$p^{\leq m} : X_{n,d}^{\leq m} \rightarrow X_{0,d}^{(m)} = \mathcal{F}_G^{\leq m}$$

is represented by the  $\mathcal{G}_n$ -torsor that parametrizes pairs  $(L, \alpha)$  where  $L$  is a lattice as above and  $\alpha : L_0/u^n L_0 \xrightarrow{\sim} L/u^n L$  is an  $R_W[[u]]/(u^n)$ -isomorphism.

Combining the above, we now see that

$$(2.5) \quad \mathcal{C}_m \simeq [X_{n,d}^{\leq m}/\phi \mathcal{G}_n],$$

where the quotient is for the action of the smooth group scheme  $\mathcal{G}_n$  on  $X_{n,d}^{\leq m}$  which is induced by  $\phi$ -conjugation. This (right) action of  $\mathcal{G}_n$  on  $X_{n,d}^{\leq m}$  can be explicitly described as follows: Let  $\gamma \in \mathcal{G}_n(R)$  which we can lift to  $g \in L^+ G(R) = GL_d(R_W[[u]])$  and consider the point  $x = (L, \alpha) \in X_{n,d}^{\leq m}(R)$  given through the matrix  $A$  by  $L = L_0 \cdot A$ ,  $\alpha = A \bmod u^n$ . (Here the elements of  $L_0$  are viewed as row vectors.) Then  $x \star \gamma$  is the point of  $X_{n,d}^{\leq m}$  that corresponds

to the matrix  $g^{-1} \cdot A \cdot \phi(g) \in LG^{\leq m}(R)$ . Observe that if  $n' > n > 2m/(p-1)$ , the natural morphism  $X_{n',d}^{\leq m} \rightarrow X_{n,d}^{\leq m}$  induces an isomorphism

$$(2.6) \quad [X_{n',d}^{\leq m} /_{\phi} \mathcal{G}_{n'}] \xrightarrow{\sim} [X_{n,d}^{\leq m} /_{\phi} \mathcal{G}_n].$$

It follows from (2.5) and the above that  $\mathcal{C}_m$  is an algebraic (Artin) stack of finite type over  $\mathbb{Z}_p$  of dimension equal to the dimension of the scheme  $\mathcal{F}_G^{\leq m}$ . It is clear that we can represent  $\mathcal{C}$  as the 2-limit of the algebraic stacks  $\mathcal{C}_m$  and so the rest of (a) follows.

For part (b) observe that the quotient description of  $\mathcal{C}_m$  implies the existence of

$$q_m : \mathcal{C}_m = [LG^{\leq m} /_{\phi} L^+ G] \rightarrow [L^+ G \setminus \mathcal{F}_G^{\leq m}] = [\mathcal{F}_G^{\leq m} / L^+ G]$$

(here in the last quotient  $g \in L^+ G$  acts by  $L \cdot g = g^{-1}L$ ). This descends the quotient morphism

$$LG^{\leq m} \rightarrow LG^{\leq m} / L^+ G = \mathcal{F}_G^{\leq m}.$$

Now for  $n \geq 2m$ ,  $U_n$  acts trivially on  $\mathcal{F}_G^{\leq m}$  and the action of  $L^+ G$  on  $\mathcal{F}_G^{\leq m}$  factors through the quotient  $\mathcal{G}_n$ . Hence composing  $q_m$  with the quotient morphism by  $U_n$ , we obtain a morphism  $q'_m : \mathcal{C}_m \rightarrow [\mathcal{F}_G^{\leq m} / \mathcal{G}_n]$ . When  $n > 2m/(p-1)$ , the morphism  $q'_m$  is given by taking the quotient

$$[X_{n,d}^{\leq m} /_{\phi} \mathcal{G}_n] \rightarrow [\mathcal{F}_G^{\leq m} / \mathcal{G}_n]$$

of the smooth torsor  $X_{n,d}^{\leq m} \rightarrow \mathcal{F}_G^{\leq m}$  and hence is smooth. It follows that  $q_m$  itself is formally smooth. Also the morphism  $\bar{q}_m$  of the statement of part (b), which is given as a composition of  $q'_m$  with  $[\mathcal{F}_G^{\leq m} / \mathcal{G}_n] \rightarrow [\mathcal{F}_G^{\leq m} / \mathcal{G}_{2m}]$ , is also smooth. A straightforward dimension count now gives that the relative dimension of  $\bar{q}_m$  is equal to the (relative) dimension of  $\mathcal{G}_{2m}$  over  $\mathbb{Z}_p$ ; this is equal to  $2md^2$ .  $\square$

2.c. We consider now some properties of  $\mathcal{R}$ ,  $\theta_m : \mathcal{C}_m \rightarrow \mathcal{R}$  and  $\theta : \mathcal{C} \rightarrow \mathcal{R}$ . Recall that, for each  $R_W((u))$ -module  $M$  which is fpqc locally on  $S = \text{Spec}(R)$  free of rank  $d$ , we have the (twisted) affine Grassmannian  $Gr_M \rightarrow S$  whose  $A$ -points for an  $R$ -algebra  $A$  are given by  $A_W[[u]]$ -lattices  $\mathfrak{M}$  of  $M_A = M \hat{\otimes}_R A$ . By [Dr] (Theorem 3.8 and Remark (b) below it),  $Gr_M$  is represented by an ind-algebraic space which is ind-proper and of ind-finite presentation over  $S$ .

**Theorem 2.5.** a) For each  $S = \text{Spec}(R) \rightarrow \mathcal{R}$  which corresponds to a  $R_W((u))$ - $\Phi$ -module  $(M, \Phi)$ , the fiber products

$$\theta \times_{\mathcal{R}} S : \mathcal{C} \times_{\mathcal{R}} S \rightarrow S, \quad \theta_m \times_{\mathcal{R}} S : \mathcal{C}_m \times_{\mathcal{R}} S \rightarrow S,$$

are represented by the (twisted) affine Grassmannian  $Gr_M \rightarrow S$ , resp. by a proper algebraic subspace of  $Gr_M \rightarrow S$ .

b) The diagonal morphism  $\delta : \mathcal{R} \rightarrow \mathcal{R} \times_{\mathbb{Z}_p} \mathcal{R}$  is representable and of finite presentation.

**Corollary 2.6.** a)  $\theta : \mathcal{C} \rightarrow \mathcal{R}$  is ind-representable and ind-proper.

b)  $\theta_m : \mathcal{C}_m \rightarrow \mathcal{R}$  is representable, proper and of finite presentation.

*Proof.* Part (a). The first part of the statement regarding  $\theta \times_{\mathcal{R}} S : \mathcal{C} \times_{\mathcal{R}} S \rightarrow S$  follows from the definition. Note here that we do not necessarily know that  $M$  contains a free  $R_W[[u]]$ -lattice. However, there is a flat base change  $R \rightarrow R'$  such that  $M' = M \hat{\otimes}_R R'$  is  $R'_W((u))$ -free. Then there is a (free)  $R'_W[[u]]$ -lattice  $\mathfrak{M}'_0$  in  $M'$ . We will now show the second part of the statement. Let  $\delta$  be the smallest integer for which

$$(2.7) \quad u^\delta \mathfrak{M}'_0 \subset \Phi(\phi^* \mathfrak{M}'_0) \subset u^{-\delta} \mathfrak{M}'_0.$$

Set  $S' = \text{Spec}(R')$  with  $R'$  as above. Suppose  $T = \text{Spec}(A)$  is an  $S$ -scheme and set  $A' = A \otimes_R R'$ ,  $T' = T \times_S S' = \text{Spec}(A')$ . Let  $\mathfrak{M}$  be an  $A'_W[[u]]$ - $\Phi$ -lattice in  $M \hat{\otimes}_R A'$  that corresponds to an object of  $(\mathcal{C}_m \times_{\mathcal{R}} S')(T')$ . For simplicity, set  $\mathfrak{M}'_{0,A} = \mathfrak{M}'_0 \hat{\otimes}_{R'} A'$ . Then

$$(2.8) \quad u^m \mathfrak{M} \subset \Phi(\phi^* \mathfrak{M}) \subset u^{-m} \mathfrak{M}, \quad u^N \mathfrak{M}'_{0,A} \subset \mathfrak{M} \subset u^{-N} \mathfrak{M}'_{0,A},$$

for some  $N \geq 0$  (we can suppose that  $N$  is the smallest integer with this property). Applying  $\Phi$  to the second chain of inclusions (2.8) gives

$$u^{pN} \Phi(\phi^* \mathfrak{M}'_{0,A}) \subset \Phi(\phi^* \mathfrak{M}) \subset u^{-pN} \Phi(\phi^* \mathfrak{M}'_{0,A})$$

and  $pN$  is the smallest integer with this property. On the other hand, we have

$$\begin{aligned} \Phi(\phi^* \mathfrak{M}) &\subset u^{-m} \mathfrak{M} \subset u^{-m-N} \mathfrak{M}'_{0,A} \subset u^{-m-N-\delta} \Phi(\phi^* \mathfrak{M}'_{0,A}), \text{ and} \\ u^{m+N+\delta} \Phi(\phi^* \mathfrak{M}'_{0,A}) &\subset u^{m+N} \mathfrak{M}'_{0,A} \subset u^m \mathfrak{M} \subset \Phi(\phi^* \mathfrak{M}). \end{aligned}$$

Combining these gives

$$u^{N+m+\delta} \Phi(\phi^* \mathfrak{M}'_{0,A}) \subset \Phi(\phi^* \mathfrak{M}) \subset u^{-N-m-\delta} \Phi(\phi^* \mathfrak{M}'_{0,A}).$$

This implies that  $pN \leq N + m + \delta$ , i.e  $N \leq (m + \delta)/(p - 1)$ , and so

$$u^{[\frac{m+\delta}{p-1}]} \mathfrak{M}'_{0,A} \subset \mathfrak{M} \subset u^{-[\frac{m+\delta}{p-1}]} \mathfrak{M}'_{0,A}.$$

(This is essentially the same argument as in [Ki1] Prop. 2.1.7.) By the above, and the definition [Dr] of the ind-structure on  $Gr_M$ , this implies that  $\mathcal{C}_m \times_{\mathcal{R}} S'$  is represented by a proper  $S'$ -scheme; therefore, by descent,  $\mathcal{C}_m \times_{\mathcal{R}} S$  is an  $S$ -proper algebraic space.

Part (b). Suppose that  $M, N$  are two  $R_W((u))$ - $\Phi$ -modules of rank  $d$ . Consider the functor on  $R$ -algebras

$$A \mapsto \text{Isom}_{\mathcal{R}}(M, N)(A) := \text{Isom}_{\Phi, A_W((u))}(M_A, N_A),$$

where for simplicity we write  $M_A = M \hat{\otimes}_R A$ ,  $N_A = N \hat{\otimes}_R A$ . We will show that this is representable by a scheme of finite presentation over  $R$ . This implies then the statement in (b). Using the existence of  $R_W$ -lattices in both  $M$  and  $N$  ([Dr], see Remark 2.4), we can see that the functor that sends  $R$  to the  $R_W((u))$ -linear isomorphisms  $M \rightarrow N$  is represented by an ind-scheme. It is not hard to see that  $\text{Isom}_{\mathcal{R}}(M, N)$  is represented by a ind-closed ind-subscheme of this ind-scheme. To show that this is actually a scheme of finite presentation we can employ an fpqc base change  $R \rightarrow R'$  and assume that  $M' = M_{R'}$ ,  $N' = N_{R'}$  are given by  $A, B \in LG(R')$ . By the definitions, the additional condition on the

$R'_W((u))$ -linear isomorphism  $M' \rightarrow N'$  given by  $g \in GL_d(R'_W((u)))$  that guarantees that it respects  $\Phi$  is

$$A = g^{-1} \cdot B \cdot \phi(g), \quad \text{or equivalently,}$$

$$(2.9) \quad g = B \cdot \phi(g) \cdot A^{-1}.$$

Suppose that  $A$  and  $B$  are in  $LG^{\leq m}(R')$ ,  $LG^{\leq n}(R')$  respectively. Assume that  $g$  is in  $LG^{\leq s}(R')$  and  $s = s(g)$  is the smallest integer with that property. Then  $\phi(g)$  belongs to  $LG^{\leq ps}(R')$  and we can see that  $ps$  is the smallest integer with this property. The identities above now imply that  $\phi(g)$  is in  $LG^{\leq s+m+n}(R')$ . Therefore  $ps \leq s + m + n$  which gives

$$(2.10) \quad s \leq \frac{m+n}{p-1}.$$

Let us write out (2.9) explicitly

$$(2.11) \quad \sum_{i \geq -s} g_i u^i = B \cdot \left( \sum_{i \geq -s} g_i u^{pi} \right) \cdot A^{-1} = \sum_{i \geq -s} u^{pi} \cdot (B \cdot g_i \cdot A^{-1})$$

with  $g_i \in M_d(R')$ . Now consider the matrix identity obtained by comparing the  $u^a$  terms of both sides of (2.11) for  $a > (m+n)/(p-1)$ . We see that this has the form

$$(2.12) \quad g_a = \sum_{i,k,l} B_l \cdot g_i \cdot A'_k$$

with  $pi + k + l = a$  and  $i \geq -s$ ,  $k \geq -m$ ,  $l \geq -n$  and  $A^{-1} = \sum_{k=-m}^{\infty} A'_k u^k$ . Since  $a > (m+n)/(p-1)$ , these inequalities imply that  $i < a$ . Therefore, all these matrix identities for  $a > (m+n)/(p-1)$  amount to determining  $g_a$  from  $g_i$  for  $i < a$ . The result now follows.  $\square$

**Corollary 2.7.** *There is a diagram*

$$\begin{array}{ccc} & \mathcal{C} & \\ \theta \swarrow & & \searrow q \\ \mathcal{R} & & [L^+ G \backslash \mathcal{F}_G], \end{array}$$

where the morphism  $q$  is formally smooth and the morphism  $\theta$  is ind-representable and ind-proper.

2.c.1. Recall that we set  $G = \text{Res}_{W/\mathbb{Z}_p} GL_d$ . Let  $K_0$  be the fraction field of  $W$  and set  $f = [K_0 : \mathbb{Q}_p] = [k : \mathbb{F}_p]$ . Then, after ordering the elements of  $\text{Gal}(K_0/\mathbb{Q}_p)$ , we can write

$$G(K_0) = \prod_{i=1}^f GL_d(K_0), \quad (\mathcal{F}_G)_W = \prod_{i=1}^f (\mathcal{F}_{GL_d})_W.$$

Let us set  $\nu = (\nu(1), \dots, \nu(f))$  where for each  $i = 1, \dots, f$ ,  $\nu(i) = (n_1(i), \dots, n_d(i))$  is a collection of integers with  $n_1(i) \geq n_2(i) \geq \dots \geq n_d(i)$ . Let  $F \subset K_0$  be the fixed field of the subgroup of  $\text{Gal}(K_0/\mathbb{Q}_p)$  that fixes  $\nu$ .

Denote by  $u^{\nu(i)}$  the diagonal matrix  $\text{diag}(u^{n_1(i)}, \dots, u^{n_d(i)})$  in  $GL_d(W((u)))$  and set

$$u^\nu = (u^{\nu(1)}, \dots, u^{\nu(d)}) \in LG(W).$$

Suppose that  $N = \max\{|n_j(i)|\}$ . Let  $S_\nu^0$ , resp.  $S_\nu$ , be the corresponding open, resp. projective, affine Schubert variety in  $(\mathcal{F}_G^{\leq N})_W \subset (\mathcal{F}_G)_W$  which is given as the image of  $L^+G \cdot u^\nu \cdot L^+G$ , resp. the Zariski closure of that image. By descent, we see that this is defined over the integers  $W'$  of the reflex field  $F$ . We set  $\mathcal{C}_\nu$  for the Artin stack over  $W'$  which is the inverse image of  $S_\nu$  under  $q$ ; this is a closed substack of  $(\mathcal{C}_N)_{W'}$ . If  $n_d(i) \geq 0$  for all  $i$ , then for  $S = \text{Spec}(R)$ , the groupoid  $\mathcal{C}_\nu(S)$  is given by  $R_W[[u]]$ - $\Phi$ -modules  $(\mathfrak{M}, \Phi)$  of rank  $d$  such that  $\Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$  and such that the action of  $u$  on  $\text{Coker}(\Phi) = \mathfrak{M}/\Phi(\phi^*\mathfrak{M})$  has elementary divisors  $u^{\nu'}$  with  $\nu'$  which satisfies  $\nu'(i) \leq \nu(i)$  in the usual ordering, for all  $i$ ;  $\mathcal{C}_\nu$  is a closed substack of  $\mathcal{C}_N$ .

The obvious version of Theorem 2.1 holds for the stack  $\mathcal{C}_\nu$ ; it is an Artin stack of finite type over  $W'$  smoothly equivalent to the Schubert variety  $S_\nu$  in the affine Grassmannian  $\mathcal{F}_G$ . Furthermore, restricting  $\theta$  to  $\mathcal{C}_\nu$  gives

$$\theta_\nu : \mathcal{C}_\nu \rightarrow \mathcal{R}_{W'} ,$$

which is representable, proper and of finite presentation. (Similarly, we can consider the inverse image  $\mathcal{C}_\nu^0$  of  $S_\nu^0$  under  $q$  and the restriction  $\theta_\nu : \mathcal{C}_\nu^0 \rightarrow \mathcal{R}_{W'}$  which is therefore representable and of finite presentation.) Summarizing, we obtain a diagram

(2.13)

$$\begin{array}{ccc} & \mathcal{C}_\nu & \\ \theta_\nu \swarrow & & \searrow q \\ \mathcal{R}_{W'} & & [(L^+G)_{W'} \setminus S_\nu] , \end{array}$$

where the morphism  $q$  is formally smooth and the morphism  $\theta_\nu$  is representable and proper.

2.d. Let us sketch how to generalize the above theory to reductive groups. Let  $H$  be any reductive algebraic group scheme  $H$  over  $W$ . Let us set  $G = \text{Res}_{W/\mathbb{Z}_p}(H)$ . Instead of  $R_W[[u]]$ - $\Phi$ -modules of rank  $d$  we consider  $H$ -torsors  $\mathcal{T}$  over  $R_W[[u]]$  together with a  $H$ -isomorphism  $\Phi : \phi_S^*(\mathcal{T}[1/u]) \xrightarrow{\sim} \mathcal{T}[1/u]$  (here  $S = \text{Spec}(R)$ ). The corresponding fpqc stack  $\mathcal{C}_G$  can be viewed as the quotient  $[LG/\phi L^+G]$ . Similarly, we can consider the fpqc stack  $\mathcal{R}_G$  of  $H$ -torsors  $T$  over  $R_W((u))$  (which are trivial fpqc locally on  $R$ ), together with a  $H$ -isomorphism  $\Phi : \phi_S^*(T) \xrightarrow{\sim} T$ . The stack  $\mathcal{R}_G$  can be viewed as the quotient  $[LG/\phi LG]$ . The obvious generalizations of Theorems 2.1 and Corollary 2.6 hold in this situation. The proofs are extensions of the above proofs for  $GL_d$  after using a faithful representation  $H \hookrightarrow GL_N$ . For example, the ind-structure is modeled on the ind-scheme structure of

$LG = \lim_m (LG \cap L\text{Res}_{W/\mathbb{Z}_p} GL_N^{\leq m})$ . In particular, we again obtain a diagram

$$(2.14) \quad \begin{array}{ccc} & \mathcal{C}_G & \\ \theta \swarrow & & \searrow q \\ \mathcal{R} & & [L^+ G \setminus \mathcal{F}_G], \end{array}$$

where the morphism  $q$  is formally smooth and the morphism  $\theta$  is ind-representable and ind-proper.

Suppose we are given a dominant coweight  $\nu$  of  $G$  defined over an unramified extension  $F$  of  $\mathbb{Q}_p$  with integers  $W'$ . Denote by  $u^\nu \in LG(F) = G(F((u)))$  the element given as the image of  $u \in \mathbb{G}_m(F((u)))$  by the corresponding homomorphism  $\mathbb{G}_m \rightarrow G_F$ . As in §2.c.1, we can define  $S_\nu^0$ , resp.  $S_\nu$ , to be the corresponding open, resp. projective, affine Schubert variety in  $(\mathcal{F}_G)_{W'}$  which is given as the image, resp. the closure of the image of  $(L^+ G)_{W'} \cdot u^\nu \cdot (L^+ G)_{W'}$ .

We set  $\mathcal{C}_{G,\nu}$  for the Artin stack over  $W'$  which can be defined by descent as the inverse image of  $S_\nu$  under  $q$ . The obvious version of Theorem 2.1 holds for the stacks  $\mathcal{C}_{G,\nu}$ ; they are Artin stacks of finite type over  $W'$  smoothly equivalent to Schubert varieties in the affine Grassmannian  $\mathcal{F}_G$ . Once again, we have a diagram

$$(2.15) \quad (\mathcal{R}_G)_{W'} \xleftarrow{\theta_\nu} \mathcal{C}_{G,\nu} \xrightarrow{q_\nu} [(L^+ G)_{W'} \setminus S_\nu],$$

where the morphism  $q_\nu$  is formally smooth and the morphism  $\theta_\nu$  is representable and proper.

### 3. $p$ -ADIC MODELS AND LOCAL MODELS

In this section, we define the stacks  $\mathcal{C}_{d,h,K}$  and prove Theorem 0.1 of the introduction. Let  $K$  a finite extension of  $\mathbb{Q}_p$  with residue field  $k$  and ramification index  $e$ . Choose a uniformizer  $\pi$  of  $K$  with Eisenstein polynomial  $E(u)$  over  $K_0 = \text{Fr}(W)$ . Then we can write  $\mathcal{O}_K \simeq W[[u]]/(E(u))$ . Fix  $h \geq 1$  (the “height”). For  $a \geq 1$ , note that  $u^{ea}$  vanishes in  $W_a[[u]]/(E(u))$  where  $W_a = W/p^a W$ . Hence, if  $R$  is a  $\mathbb{Z}/p^a\mathbb{Z}$ -algebra, we have

$$(3.16) \quad u^{ea} R_W[[u]] \subset E(u)^h R_W[[u]].$$

Now consider the category  $\text{Nil}_p$  of schemes  $S$  such that  $p^b \cdot \mathcal{O}_S = 0$  for some  $b \geq 1$ . Such schemes can be viewed as formal schemes over  $\mathbb{Z}_p$ . We will call set-valued functors on  $\text{Nil}_p$  which satisfy descent for the fpqc topology “formal spaces”. A formal scheme  $X$  over  $\text{Spf}(\mathbb{Z}_p)$  gives a formal space by sending  $S$  in  $\text{Nil}_p$  to the set of formal scheme morphisms  $S \rightarrow X$  over  $\text{Spf}(\mathbb{Z}_p)$ . Also if  $\mathcal{S}$  is a fpqc stack over  $\mathbb{Z}_p$ , we can consider the restriction  $\hat{\mathcal{S}}$  to a groupoid over the category  $\text{Nil}_p$ ; we can think of the “formal stack”  $\hat{\mathcal{S}}$  as “the formal completion of  $\mathcal{S}$  along its fiber over  $p$ ”.

3.a. Consider the functor  $M_{d,h,K}$  on schemes over  $\mathbb{Z}_p$  that associates to  $S = \text{Spec}(R)$  the set of  $R_W[u]$ -submodules

$$(3.17) \quad \mathcal{E} \subset (R_W[u]/(E(u)^h))^d,$$

such that both  $\mathcal{E}$  and the quotient  $(R_W[u]/(E(u)^h))^d/\mathcal{E}$  are  $R_W$ -projective with rank locally constant on  $\text{Spec}(R)$ . This functor is represented by a projective scheme over  $\mathbb{Z}_p$  (a disjoint sum of closed subschemes of Grassmannians), which we will also denote by  $M_{h,K}$ . Once again, here and in what follows we will omit the subscript  $d$  from the notation. In fact, if in addition  $h = 1$ , we will also omit  $h$  from the notation and simply write  $M_K$ . The group scheme

$$\text{Res}_{(W[u]/(E(u)^h))/\mathbb{Z}_p} GL_d$$

over  $\mathbb{Z}_p$  acts on  $M_{h,K}$ .

Suppose that  $p^a \cdot R = 0$  for  $a \geq 1$ . Then  $M_{h,K}(R)$  is in bijection with the set of  $R_W[[u]]$ -modules  $\mathcal{L}$  with

$$u^{eah} \cdot R_W[[u]]^d \subset E(u)^h R_W[[u]]^d \subset \mathcal{L} \subset R_W[[u]]^d$$

which are, locally on  $R$ , free over  $R_W[[u]]$ . This gives a functorial injection,

$$(3.18) \quad M_{h,K}(R) \hookrightarrow \mathcal{F}_G(R)$$

and it implies that we can view the formal completion  $\hat{M}_{h,K}$  of  $M_{h,K}$  along its fiber over  $p$  as a subspace of the formal space  $\hat{\mathcal{F}}_G$  defined by the affine Grassmannian  $\mathcal{F}_G$ .

3.b. We now define a groupoid  $\mathcal{C}_{d,h,K}$  over  $\mathbb{Z}_p$ -schemes as follows. Let  $R$  be a  $\mathbb{Z}_p$ -algebra. Then  $\mathcal{C}_{d,h,K}(R)$  is given by pairs  $(\mathfrak{M}, \Phi)$  of an  $R_W[[u]]$ -module  $\mathfrak{M}$  which is, locally fpqc on  $\text{Spec}(R)$ , free of rank  $d$  and a  $R_W((u))$ -module isomorphism

$$(3.19) \quad \Phi : \phi^* \mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u],$$

such that  $E(u)^h \mathfrak{M} \subset \Phi(\phi^* \mathfrak{M}) \subset \mathfrak{M}$ . We can see that the groupoid  $\mathcal{C}_{d,h,K}$  is an fpqc stack. In what follows, we will omit  $d$  from the notation and write  $\mathcal{C}_{h,K}$ .

We can consider the formal  $p$ -adic completion  $\hat{\mathcal{C}}_{h,K}$  of  $\mathcal{C}_{h,K}$ . This is a fpqc stack over  $\text{Nil}_p$  defined by considering  $\mathcal{C}_{h,K}(R)$  as above for  $\mathbb{Z}_p$ -algebras  $R$  in which  $p$  is nilpotent. We can write  $\hat{\mathcal{C}}_{h,K}$  as a 2-limit

$$\hat{\mathcal{C}}_{h,K} := \lim_{\xrightarrow{a}} \mathcal{C}_{h,K}^a$$

where  $\mathcal{C}_{h,K}^a := \hat{\mathcal{C}}_{h,K} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a \mathbb{Z}$  is the reduction modulo  $p^a$ . Using (3.16) we can see that  $\mathcal{C}_{h,K}^a$  is a (closed) substack of  $\mathcal{C}_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a \mathbb{Z}$ . Now suppose that  $\mathcal{X}$  is a formal scheme over  $\text{Spf}(\mathbb{Z}_p)$  such that  $p\mathcal{O}_{\mathcal{X}}$  is an ideal of definition. Then, for each  $a \geq 1$ ,  $\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a \mathbb{Z}$  is a scheme over  $\mathbb{Z}/p^a \mathbb{Z}$ . We set

$$\hat{\mathcal{C}}_{h,K}(\mathcal{X}) := \lim_{\xleftarrow{a}} \hat{\mathcal{C}}_{h,K}(\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a \mathbb{Z}).$$

This allows us to extend  $\widehat{\mathcal{C}}_{h,K}$  to a groupoid over the category of adic formal schemes over  $\mathrm{Spf}(\mathbb{Z}_p)$ . Suppose that  $R$  is a Noetherian  $p$ -adic ring (i.e a Noetherian  $\mathbb{Z}_p$ -algebra which is  $p$ -adically complete and separated,  $R = \lim_{\leftarrow a} R/p^a R$ ), and set  $\mathcal{X} = \mathrm{Spf}(R)$ . Set

$$R_W\{\{u\}\} = \lim_{\leftarrow a} [(R_W/p^a R_W)((u))] = \left\{ \sum_{i=-\infty}^{+\infty} a_i u^i \mid a_i \in R_W, \lim_{i \rightarrow -\infty} a_i = 0 \right\}.$$

We can see that the objects of  $\widehat{\mathcal{C}}_{h,K}(\mathrm{Spf}(R))$  are given by pairs  $(\mathfrak{M}, \Phi)$  of an  $R_W[[u]]$ -module  $\mathfrak{M}$  which is locally  $R_W[[u]]$ -free of rank  $d$  and a  $R_W\{\{u\}\}$ -module isomorphism

$$(3.20) \quad \Phi : \phi^* \mathfrak{M} \otimes_{R_W[[u]]} R_W\{\{u\}\} \xrightarrow{\sim} \mathfrak{M} \otimes_{R_W[[u]]} R_W\{\{u\}\},$$

such that

$$(3.21) \quad E(u)^h \mathfrak{M} \subset \Phi(\phi^* \mathfrak{M}) \subset \mathfrak{M}.$$

(Note that  $E(u)$  is a unit in  $R_W\{\{u\}\}$  and so therefore also in  $R_W\{\{u\}\}$ .)

3.b.1. For  $\mathrm{Spec}(R)$  in  $\mathrm{Nil}_p$  now set

$$LG^{h,K}(R) = \{A \in M_d(R_W[[u]]) \mid A^{-1} \in E(u)^{-h} \cdot M_d(R_W[[u]]) \subset M_d(R_W((u)))\}.$$

This defines a functor on  $\mathrm{Nil}_p$ . As before, we can write

$$\widehat{\mathcal{C}}_{h,K} = [LG^{h,K}/_\phi L^+ G].$$

where (by abusing notation) we also denote by  $L^+ G$  the formal  $p$ -adic completion of  $L^+ G$ .

The map

$$A \mapsto A \cdot R_W[[u]]^d \subset R_W((u))^d$$

gives a morphism of formal stacks,

$$q_{h,K} : \widehat{\mathcal{C}}_{h,K} = [LG^{h,K}/_\phi L^+ G] \rightarrow [L^+ G \setminus \widehat{M}_{h,K}].$$

3.b.2. Using Proposition 2.2 and the above, we see that if  $n(a) > eah/(p-1)$  then

$$[LG^{h,K}/_\phi U_{n(a)}]_{\mathbb{Z}/p^a \mathbb{Z}} \simeq [LG^{h,K}/U_{n(a)}]_{\mathbb{Z}/p^a \mathbb{Z}}.$$

As in the proof of Theorem 2.1 we can see that the quotient stack  $[LG^{h,K}/U_{n(a)}]_{\mathbb{Z}/p^a \mathbb{Z}}$  is represented by a torsor  $(X_{n(a),d}^{h,K})_{\mathbb{Z}/p^a \mathbb{Z}}$  for the group scheme  $(L^+ G/U_{n(a)})_{\mathbb{Z}/p^a \mathbb{Z}} = (\mathcal{G}_{n(a)})_{\mathbb{Z}/p^a \mathbb{Z}}$  over  $(M_{h,K})_{\mathbb{Z}/p^a \mathbb{Z}}$ . Similarly to that proof, we can conclude

$$\mathcal{C}_{h,K}^a \simeq [(X_{n(a),d}^{h,K})_{\mathbb{Z}/p^a \mathbb{Z}}/_\phi (\mathcal{G}_{n(a)})_{\mathbb{Z}/p^a \mathbb{Z}}],$$

and that the morphism

$$q_{h,K}^a : \mathcal{C}_{h,K}^a \rightarrow [L^+ G \setminus M_{h,K}]_{\mathbb{Z}/p^a \mathbb{Z}}$$

is formally smooth. Hence, the morphism between the formal stacks

$$(3.22) \quad q_{h,K} : \widehat{\mathcal{C}}_{h,K} \rightarrow [L^+ G \setminus \widehat{M}_{h,K}]$$

is also formally smooth.

Recall the definition of the stack  $\mathcal{R} = \mathcal{R}_d$  over  $\mathbb{Z}_p$ -schemes, cf. §2.a, and the notations  $\mathcal{R}^a = \mathcal{R} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$ ,  $\mathcal{C}_{h,K}^a = \mathcal{C}_{h,K} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$ . By sending an object  $(\mathfrak{M}, \Phi)$  to  $(\mathfrak{M}[1/u], \Phi)$ , we obtain a morphism of stacks  $\theta : \mathcal{C}_{h,K} \rightarrow \mathcal{R}$ . Note that the morphism  $\widehat{\theta} : \widehat{\mathcal{C}}_{h,K} \rightarrow \widehat{\mathcal{R}}$  on formal completions is obtained by passing to the limit on the morphisms  $\theta^a : \mathcal{C}_{h,K}^a \rightarrow \mathcal{R}^a$  which arise by restricting the morphisms

$$\theta_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z} : \mathcal{C}_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z} \rightarrow \mathcal{R}^a$$

to the closed substacks  $\mathcal{C}_{h,K}^a$ . This together with Corollary 2.6 implies that the morphisms  $\theta^a$  are representable and proper.

Our discussion, in the previous two paragraphs gives the proof of Theorem 0.1 of the introduction when  $h = 1$ .

**Remark 3.1.** Of course, the above actually shows that the obvious generalization of Theorem 0.1 to  $\mathcal{C}_{h,d,K}$  for any  $h \geq 1$  is also valid.

**Remark 3.2.** a) The morphism  $\theta : \mathcal{C}_{h,K} \rightarrow \mathcal{R}$  is ind-representable: Indeed, suppose that we are given a point  $\xi : S = \text{Spec}(R) \rightarrow \mathcal{R}$  corresponding to a module  $(M, \Phi)$  over  $R_W((u))$ . Then by Theorem 2.5, the fiber  $\mathcal{C} \times_{\mathcal{R}, \xi} S \rightarrow S$  is ind-represented by an ind-algebraic space. We can see that the subspace  $\mathcal{C}_{h,K} \times_{\mathcal{R}, \xi} S \hookrightarrow \mathcal{C} \times_{\mathcal{R}, \xi} S$  is described by a closed condition and it is a closed ind-algebraic subspace.

b) In general,  $\mathcal{C}_{h,K} \times_{\mathcal{R}, \xi} S \rightarrow S$  is not representable for all  $S = \text{Spec}(R)$ . However, assume that  $R \simeq \lim_{\leftarrow a} R/p^a R$  is a Noetherian  $p$ -adic ring and that

$$\hat{\xi} = (\xi^a), \quad \xi^a : \text{Spec}(R/p^a R) \rightarrow \mathcal{R}^a,$$

is a point of the formal completion  $\widehat{\mathcal{R}}$ . Assume also that the  $R_W(\{u\})$ -module  $\hat{M}$  which corresponds to  $\hat{\xi}$  is free over  $R_W(\{u\})$ . Fix a basis  $R_W(\{u\})^d \simeq \hat{M}$  and set  $M = R_W((u))^d \subset \hat{M}$ . The affine Grassmannian  $Gr_M \rightarrow S$  is ind-projective and supports a natural line bundle whose restriction on each closed subscheme is very ample. As above, we can see that for each  $a$ , the fiber of the morphism  $\theta^a : \mathcal{C}_{h,K}^a \rightarrow \mathcal{R}^a$  over  $\xi^a$  is representable by a closed (and hence) projective subscheme of  $Gr_M \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$ . Varying  $a$ , this defines a formal scheme over  $\text{Spf}(R)$ . As in [Ki3], proof of Prop. 1.3, by using the above ample line bundle on  $Gr_M$ , we may algebraicize this  $p$ -adic formal scheme over  $\text{Spf}(R)$  to a projective scheme  $\mathcal{C}_{K,\xi}$  over  $\text{Spec}(R)$ . The result is that, in this case, the fiber  $\widehat{\mathcal{C}}_{d,h,K} \times_{\widehat{\mathcal{R}}, \hat{\xi}} \hat{S} \rightarrow \hat{S}$  between the formal completions is representable by the formal scheme over  $\hat{S} = \text{Spf}(R)$  associated to the projective scheme  $\mathcal{C}_{K,\xi} \rightarrow \text{Spec}(R)$ . If in addition to the hypotheses above,  $R/pR$  is of finite type over  $\mathbb{F}_p$ , the arguments in the proof of [Ki3] Prop. 1.6.4, show that the morphism  $\mathcal{C}_{K,\xi} \rightarrow \text{Spec}(R)$  induces a closed immersion between the generic fibers.

3.c. Choose a cocharacter

$$\mu : \bar{\mathbb{Q}}_p^\times \rightarrow (\text{Res}_{K/\mathbb{Q}_p} GL_d)(\bar{\mathbb{Q}}_p) = \prod_{\psi: K \rightarrow \bar{\mathbb{Q}}_p} GL_d(\bar{\mathbb{Q}}_p)$$

defined over  $\bar{\mathbb{Q}}_p$  whose conjugacy class is defined over the reflex field  $E$ . The projection to the component corresponding to  $\psi$

$$\mu_\psi = pr_\psi \circ \mu : \bar{\mathbb{Q}}_p^\times \rightarrow GL_d(\bar{\mathbb{Q}}_p)$$

provides a grading

$$(3.23) \quad \bar{\mathbb{Q}}_p^d = \bigoplus_{n \in \mathbb{Z}} V_n^\psi ,$$

with  $V_n^\psi = \{v \in \bar{\mathbb{Q}}_p^d \mid \mu_\psi(a) = a^n v\}$ . Let  $h_+$ , resp.  $h_-$ , the maximum, resp. minimum value of  $n$  (among all the values for all  $\psi$ ) for which  $V_n^\psi \neq (0)$ . Set  $h = h_+ - h_-$ .

We will now define the corresponding local model  $M_{\mu,K}^{\text{loc}}$  over  $\mathcal{O}_E$ ; it is going to be a projective subscheme of  $M_{d,h,K}$  (see 3.a).

First, we define the generic fiber of  $M_{\mu,K}^{\text{loc}}$  over the reflex field  $E$ . Suppose that  $R$  is a  $\bar{\mathbb{Q}}_p$ -algebra, and fix an embedding  $\psi : K \rightarrow \bar{\mathbb{Q}}_p$ , this induces a homomorphism

$$R_W = W \otimes_{\mathbb{Z}_p} R \rightarrow R ; \quad a \otimes r \mapsto \psi(a)r .$$

Elements of  $M_{h,K}(R)$  correspond bijectively to  $R_W[u]$ -modules  $\mathcal{M}$  such that

$$E(u)^{h_+} R_W[u]^d \subset \mathcal{M} \subset E(u)^{h_-} R_W[u]^d$$

with graded pieces  $R_W$ -projective and with rank locally constant on  $\text{Spec}(R)$ . Write

$$\text{Norm}_{K_0/\bar{\mathbb{Q}}_p}(E(u)) = \prod_{\psi: K \rightarrow \bar{\mathbb{Q}}_p} (u - \varpi_\psi) \in \bar{\mathbb{Q}}_p[u]$$

so that  $\varpi_\psi = \psi(\pi)$ . Using this, we see that we can write  $\mathcal{M} = \bigoplus_\psi \mathcal{M}_\psi$  with  $\mathcal{M}_\psi$  a  $R[u]$ -submodule with

$$(u - \varpi_\psi)^{h_+} R[u]^d \subset \mathcal{M}_\psi \subset (u - \varpi_\psi)^{h_-} R[u]^d .$$

For each such  $\psi$ , consider the  $R$ -module

$$\mathcal{M}_\psi \cap (u - \varpi_\psi)^j R[u]^d / \mathcal{M}_\psi \cap (u - \varpi_\psi)^{j+1} R[u]^d$$

We ask that for each  $\psi$ ,  $j \in \mathbb{Z}$ , this is a projective  $R$ -module of rank  $\dim(V_j^\psi)$ . We can see that this condition defines a locally closed subvariety of  $M_{h,K} \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$ . This carries an action of  $\text{Gal}(\bar{\mathbb{Q}}_p/E)$  that allows us to descend it to a subvariety  $Z$  of  $M_{h,K} \otimes_{\mathbb{Z}_p} E$ . By definition, the generic fiber of the local model  $M_{\mu,K}^{\text{loc}}$  is the Zariski closure  $\bar{Z}$  of  $Z$  in  $M_{h,K} \otimes_{\mathbb{Z}_p} E$ . Finally, by definition, the local model  $M_{\mu,K}^{\text{loc}}$  is the flat closure of  $\bar{Z}$  in  $M_{h,K} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ .

Observe that, by the above, for each  $\mathcal{O}_E$ -scheme  $S = \text{Spec}(R)$  in  $\text{Nil}_p$  we have

$$(3.24) \quad M_{\mu,K}^{\text{loc}}(R) \hookrightarrow (\mathcal{F}_G \otimes_{\mathbb{Z}_p} \mathcal{O}_E)(R).$$

**Remark 3.3.** Suppose that, for all  $\psi$ , we have  $n \in \{0, 1\}$  in (3.23). Then  $\mu$  is minuscule. Assume  $h_+ = 1$ ,  $h_- = 0$ , which is the typical case. Then  $h = 1$ ,  $R_W[u]/(E(u)^h) = \mathcal{O}_K \otimes_{\mathbb{Z}_p} R$ , and  $M_{h,K}(R)$  is given by  $\mathcal{O}_K \otimes_{\mathbb{Z}_p} R$ -submodules

$$\mathcal{E} \subset (\mathcal{O}_K \otimes_{\mathbb{Z}_p} R)^d$$

which are locally on  $R$  direct summands as  $R_W$ -modules. Set  $r_\psi = \dim(V_0^\psi)$ . The conditions above amount to asking that  $\text{rank}_R(\mathcal{E}_\psi) = r_\psi$  and so  $M_{\mu,K}^{\text{loc}}$  agrees with the local model of [PR1]. If  $k = \mathbb{F}_p$ , the special fiber  $M_{\mu,K}^{\text{loc}} \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}_p$  can be identified with the affine Schubert variety  $S_\nu$  in the affine Grassmannian of  $GL_d$ , where the coweight  $\nu$  is the dual partition to  $(r_\psi)_\psi$ , i.e.,  $\nu_i = \#\{\psi \mid r_\psi \geq i\}$ , cf. [PR1], Thm. 5.4.

3.d. We continue with the above notations of §3.a. By definition we have a closed immersion

$$\widehat{M}_{\mu,K}^{\text{loc}} \hookrightarrow \widehat{M}_{h,K}$$

which is equivariant for the natural action of (the formal completion of)  $L^+G$ . Using descent and (3.22) we obtain a fpqc stack  $\widehat{\mathcal{C}}_{\mu,K}$  over  $\text{Nil}_p \cap (\text{Sch}/\mathcal{O}_E)$  together with a formally smooth morphism

$$q_{\mu,K} : \widehat{\mathcal{C}}_{\mu,K} \rightarrow [L^+G \backslash \widehat{M}_{\mu,K}^{\text{loc}}].$$

Indeed, if  $S = \text{Spec}(R)$  is an  $\text{Spec}(\mathcal{O}_E)$ -scheme in  $\text{Nil}_p$  then  $\widehat{\mathcal{C}}_{\mu,K}(R)$  is the groupoid of pairs  $(\mathfrak{M}, \Phi)$  of a  $R_W[[u]]$ -module  $\mathfrak{M}$  which is  $R_W[[u]]$ -free of rank  $d$  (fpqc) locally on  $R$  and a  $R_W((u))$ -module isomorphism

$$(3.25) \quad \Phi : \phi^*\mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u],$$

such that, locally, there is an isomorphism  $\alpha : R_W[[u]]^d \xrightarrow{\sim} \mathfrak{M}$  for which the  $R_W[[u]]$ -lattice  $\alpha^{-1}(\Phi(\phi^*\mathfrak{M})) \subset R_W((u))^d$  belongs to the subset  $\widehat{M}_{\mu,K}^{\text{loc}}(R)$  of  $(\mathcal{F}_G \otimes_{\mathbb{Z}_p} \mathcal{O}_E)(R)$ . As in §3.b.2, we see that there is also a morphism of formal completions

$$(3.26) \quad \widehat{\theta}_\mu : \widehat{\mathcal{C}}_{\mu,K} \rightarrow \widehat{\mathcal{R}}_d \otimes_{\mathbb{Z}_p} \mathcal{O}_E$$

which can be obtained as the limit of representable and proper morphisms.

#### 4. DEFORMATIONS OF GALOIS REPRESENTATIONS

In this section, we explain an aspect of the connection between the spaces of  $\Phi$ -modules and the deformation theory of Galois representations as developed by Kisin [Ki1], [Ki3]. We restrict attention to the flat or Barsotti-Tate case (cf. [Ki1]) This corresponds to the case  $h = 1$ . For simplicity, we also assume  $p$  is odd.

4.a. **Galois representations.** Suppose that  $R = \Lambda$  is a  $\mathbb{Z}_p$ -algebra with finitely many elements. As in §1 (see also [Fo]), a pair  $(M, \Phi)$  corresponding to an object of  $\mathcal{R}(\Lambda)$  gives an étale  $\Lambda$ -sheaf over  $\text{Spec}(k((u)))$  which is free of rank  $d$ , i.e. an equivalence class of a representation

$$\rho_{(M,\Phi)} : \text{Gal}(k((u))^{\text{sep}}/k((u))) \rightarrow GL_d(\Lambda).$$

As a result, an object  $(\mathfrak{M}, \Phi)$  of  $\mathcal{C}_d(\Lambda)$  also gives a representation  $\rho_{(\mathfrak{M}[1/u], \Phi)}$  of the Galois group  $\text{Gal}(k((u))^{\text{sep}}/k((u)))$ .

Now let  $\mathbb{F}$  be a finite field and suppose that

$$\rho : \text{Gal}(k((u))^{\text{sep}}/k((u))) \rightarrow GL_d(\mathbb{F})$$

is a representation which corresponds to a pair  $(M_0, \Phi_0)$ , and consider the corresponding object  $[\rho] : \text{Spec}(\mathbb{F}) \rightarrow \mathcal{R}$ . Denote by  $\mathcal{R}_{[\rho]}$  the groupoid over finite local Artinian  $\mathbb{Z}_p$ -algebras  $\Lambda$  with residue field  $\mathbb{F}$ , with objects

$$\mathcal{R}_{[\rho]}(\Lambda) = \{(M, \Phi) \in \mathcal{R}(\Lambda), \alpha : (M, \Phi) \otimes_{\Lambda} \mathbb{F} \xrightarrow{\sim} (M_0, \Phi_0)\}$$

and obvious morphisms. By the above,  $\mathcal{R}_{[\rho]}$  is identified with the groupoid of deformations  $\mathfrak{D}_{\rho}$  of the Galois representation  $\rho$ .

**4.b. Finite flat group schemes.** Now let  $K$  be a finite extension of  $\mathbb{Q}_p$  with residue field  $k$  and ramification index  $e$ . Choose a uniformizer  $\pi$  of  $K$  with Eisenstein polynomial  $E(u)$  over  $\text{Fr}(W)$ . Set  $K_{\infty} = \cup_n K(\pi_n)$ , where  $\pi_n = \pi^{1/p^n}$  are compatible choices of roots; then the theory of norm fields allows us to identify the Galois groups

$$G_{\infty} := \text{Gal}(\bar{K}/K_{\infty}) \xrightarrow{\sim} \text{Gal}(k((u))^{\text{sep}}/k((u))) ,$$

comp. [Ki1], §1. The following can be derived from [Ki2] Theorem 0.5 by taking into account the functoriality of the  $\Lambda$ -action and the properties of the Breuil-Kisin module functors (for example see [Ki1] §1.2).

**Theorem 4.1.** (Kisin) Assume  $p > 2$  and let  $\Lambda$  be a  $\mathbb{Z}_p$ -algebra with finitely many elements. There is an equivalence between the groupoid of finite flat commutative group schemes  $\mathcal{G}$  with an action of  $\Lambda$  (i.e. “ $\Lambda$ -module schemes”) over  $\mathcal{O}_K$  such that  $\mathcal{G}(\bar{\mathcal{O}}_K) \simeq \Lambda^d$ , and the groupoid of pairs  $(\mathfrak{M}, \Phi)$  of  $\Lambda_W[[u]]$ - $\Phi$ -modules with the following properties:

- a)  $\text{Coker}(\Phi)$  is annihilated by  $E(u)$ ,
- b)  $\mathfrak{M}[1/u]$  is  $\Lambda_W((u))$ -free of rank  $d$ ,
- c)  $\mathfrak{M}$  is a  $W[[u]]$ -module of projective dimension 1, i.e., equivalently by [Ki2], Lemma (2.3.2),  $\mathfrak{M}$  is an iterated extension of free  $k[[u]]$ -modules.

Under this equivalence, the restriction of

$$\rho_{\mathcal{G}} : \text{Gal}(\bar{K}/K) \rightarrow \text{Aut}_{\Lambda}(\mathcal{G}(\bar{\mathcal{O}}_K))$$

to  $G_{\infty} \simeq \text{Gal}(k((u))^{\text{sep}}/k((u)))$  is isomorphic to  $\rho_{(\mathfrak{M}[1/u], \Phi)}(1)$  [twist by the cyclotomic character].

Note that property (c) is automatically satisfied when  $p \cdot \Lambda = (0)$ . We will also consider the groupoid of modules as above with the additional property

- d)  $\mathfrak{M}$  is  $\Lambda_W[[u]]$ -free.

**Lemma 4.2.** Let  $A$  be a local Noetherian ring with residue field  $l$  and suppose that  $M \subset A((u))^d$  is a finitely generated  $A[[u]]$ -module such that  $M[1/u] = A((u))^d$  and  $M \otimes_A l \simeq l[[u]]^d$ . Then  $M \simeq A[[u]]^d$ .

*Proof.* By Nakayama's lemma, there is a surjective  $A[[u]]$ -homomorphism  $\phi : A[[u]]^d \rightarrow M$ . Then  $\phi[1/u] : A((u))^d \rightarrow M[1/u] = A((u))^d$  is also a surjection which then has to be bijective. This implies that  $\phi$  is also injective.  $\square$

Write  $\Lambda \otimes_{\mathbb{Z}_p} W = \prod_j \Lambda_j$  with  $\Lambda_j$  Artin local. An application of the lemma to  $\Lambda_j$  shows that the additional property (d) is satisfied if and only if  $\mathfrak{M} \otimes_{\Lambda} \mathbb{F} \simeq (\mathbb{F} \otimes k)[[u]]^d$ .

4.c. Suppose that  $(A, \mathfrak{m})$  is an Artin local Noetherian ring with finite residue field  $\mathbb{F}$ . A representation

$$\rho : \text{Gal}(\bar{K}/K) \rightarrow GL_d(A)$$

is called *flat* [Ra] if the corresponding  $\mathbb{Z}_p[\text{Gal}(\bar{K}/K)]$ -module is isomorphic to the twist by  $(-1)$  of the module obtained by the Galois action on the  $\mathbb{Z}_p$ -module of  $\mathcal{O}_{\bar{K}}$ -points of some commutative finite flat group scheme over  $\mathcal{O}_K$ . This notion extends to the more general situation that  $(A, \mathfrak{m})$  is a complete local Noetherian ring with finite residue field  $\mathbb{F}$ . In this case, the representation

$$\rho : \text{Gal}(\bar{K}/K) \rightarrow GL_d(A)$$

is *flat* iff, for all  $n \geq 1$ , the representation obtained by reducing  $\rho$  modulo  $\mathfrak{m}^n$  is flat, cf. [Ra].

Consider the morphism of formal stacks  $\theta_K : \widehat{\mathcal{C}}_K := \widehat{\mathcal{C}}_{1,K} \rightarrow \widehat{\mathcal{R}}$ . Suppose that  $(A, \mathfrak{m})$  is a complete local Noetherian ring with finite residue field  $\mathbb{F}$ . Let  $\xi = (\xi_n)_{n \geq 1} \in \widehat{\mathcal{R}}(A)$  be an  $A$ -valued object of  $\widehat{\mathcal{R}}$ , where for each  $n \geq 1$ ,  $\xi_n$  is in  $\widehat{\mathcal{R}}(A/\mathfrak{m}^n)$ . The 2-fiber product  $\xi_n \times_{\widehat{\mathcal{R}}} \widehat{\mathcal{C}}_K$  is representable by a projective scheme  $\mathcal{C}_{K,\xi_n} \rightarrow \text{Spec}(A/\mathfrak{m}^n)$ . In the limit, we obtain a formal scheme  $\mathcal{C}_{K,\xi}$  over  $\text{Spf}(A)$ . The argument in [Ki3], Cor. 1.5.1 (or see Remark 3.2 (b)) shows that this is algebraizable to a projective scheme

$$\mathcal{C}_{K,\xi} \rightarrow \text{Spec}(A).$$

Denote by  $A^K$  the quotient of  $A$  that corresponds to the scheme theoretic image of this morphism. We obtain

$$\xi_K : \text{Spec}(A^K) \rightarrow \text{Spec}(A) \rightarrow \widehat{\mathcal{R}}.$$

**Proposition 4.3.** *With the above notations, assume in addition that  $A$  is Artinian. Then*

$$\rho_{\xi_K} : G_{\infty} = \text{Gal}(k((u))^{\text{sep}}/k((u))) \rightarrow GL_d(A^K)$$

*extends to a representation of  $\text{Gal}(\bar{K}/K)$  which is flat.*

*Proof.* For simplicity, set  $B = A^K$ . Then there is  $B \hookrightarrow B'$  with  $B'$  a  $B$ -algebra of finite type such that  $\mathcal{C}_K$  affords a  $B'$ -valued point  $\zeta'$  that lifts  $\xi_B$ . Denote by  $(M_B, \Phi)$  the  $B_W((u))$ -module that corresponds to  $\xi_B$ . Since  $B$  is Artinian,  $M_B \simeq B_W((u))^d$ . Giving the point  $\zeta'$  amounts to giving a  $B'_W[[u]]$ -projective module  $\mathfrak{M}'$  of rank  $d$  in  $M' = M_B \hat{\otimes}_B B'$  which satisfies

$$E(u)\mathfrak{M}' \subset \Phi(\phi^*\mathfrak{M}') \subset \mathfrak{M}'.$$

Now set  $\mathfrak{M} := \mathfrak{M}' \cap M \subset M'$ ; this is a  $B_W[[u]]$ -module which is  $\Phi$ -stable. The proof of [Ki1] Prop. 2.1.4 applies to show that  $\mathfrak{M}$  satisfies properties (a), (b) and (c) (with  $\Lambda = B$ ). Therefore, the  $\mathbb{Z}_p[G_\infty]$ -module given by  $\rho(1)$  is given by a finite flat group scheme over  $\mathcal{O}_K$  as desired. (However, note here that, as pointed out by Kisin,  $\mathfrak{M}$  does not have to be  $B_W[[u]]$ -free.)  $\square$

4.c.1. Assume now that  $(A, \mathfrak{m})$  is a complete local Noetherian ring with finite residue field  $\mathbb{F}$  and that in addition:

- 1) the  $A$ -valued representation  $\rho_\xi$  of  $G_\infty$  which corresponds to  $\xi$  extends to a representation of  $\text{Gal}(\bar{K}/K)$ , and
- 2) the  $\mathbb{F}$ -valued representation of  $\text{Gal}(\bar{K}/K)$  which is obtained by reducing  $\rho_\xi$  modulo  $\mathfrak{m}$  is flat.

By [Ra] there is a quotient  $A \rightarrow A^{\text{fl}}$  such that for each  $A \rightarrow B$  with  $B$  a local Artinian  $A$ -algebra with residue field  $\mathbb{F}$ , the representation obtained by composing with  $GL_d(A) \rightarrow GL_d(B)$  is flat if and only if  $A \rightarrow B$  factors as  $A \rightarrow A^{\text{fl}} \rightarrow B$ . Using the above proposition one can see that  $A \rightarrow A^K$  factors as

$$A \rightarrow A^{\text{fl}} \rightarrow A^K.$$

**Remark 4.4.** In general, it is not clear whether we should expect that  $A^{\text{fl}} \rightarrow A^K$  is an isomorphism. The issue is the following: Consider a deformation  $\rho$  of  $\rho_0 = \rho_\xi$  modulo  $\mathfrak{m}$  over a finite Artin local  $\mathbb{Z}_p$ -algebra  $\Lambda$  with residue field  $\mathbb{F}$ . It corresponds to a  $\Lambda_W((u))$ - $\Phi$ -module  $M$ . By assumption, the  $(\mathbb{F} \otimes_{\mathbb{Z}_p} W)((u))$ - $\Phi$ -module  $M_0 = M \otimes_{\Lambda} \mathbb{F}$  which corresponds to  $\rho_0$  contains a  $(\mathbb{F} \otimes_{\mathbb{Z}_p} W)[[u]]$ - $\Phi$ -submodule  $\mathfrak{M}_0$  with  $E(u)\mathfrak{M}_0 \subset \Phi(\phi^*\mathfrak{M}_0) \subset \mathfrak{M}_0$ . We can easily see that  $\mathfrak{M}_0 \simeq (\mathbb{F} \otimes_{\mathbb{Z}_p} W)[[u]]^d$ . Assume now that  $\rho$  is also flat; this implies that  $M$  contains a  $\Lambda_W[[u]]$ - $\Phi$ -module  $\mathfrak{M}$  with  $\mathfrak{M}[1/u] = M$  that satisfies properties (a), (b) and (c). The problem is that if  $e > p - 1$ , we cannot expect that  $\mathfrak{M}$  is a deformation of  $\mathfrak{M}_0$  (so that we can apply Lemma 4.2). The question is: Is there is some  $\mathbb{Z}_p$ -algebra  $C$  that contains  $\Lambda$  and a  $C_W[[u]]$ - $\Phi$ -module  $\mathfrak{M}_C$  with  $\mathfrak{M}_C[1/u] = M \otimes_{\Lambda} C$  which is in addition  $C_W[[u]]$ -projective and such that  $\mathfrak{M} = M \cap \mathfrak{M}_C$ ? Our discussion implies

**Proposition 4.5.** *In the situation of Remark 4.4, assume that  $e \leq p - 1$ . Then  $A^{\text{fl}} \simeq A^K$ .*

## 5. COEFFICIENT DOMAINS AND A PERIOD MORPHISM

5.a. Fix  $d$ , the local field  $K$  and  $h \geq 1$ . We define the stack in groupoids  $\mathcal{D}_{d,h,K}$  over schemes over  $\mathbb{Q}_p$  which is described as follows:

If  $R$  is a  $\mathbb{Q}_p$ -algebra, then the objects of  $\mathcal{D}_{d,h,K}(R)$  are triples  $(D, \Phi, \text{Fil}^\bullet)$  where

- $D$  is a  $R \otimes_{\mathbb{Q}_p} K_0$ -module which is, locally on  $R$ , free of rank  $d$ ,
- $\Phi : D \rightarrow D$  is an  $\text{Id} \otimes_{\mathbb{Q}_p} \phi$ -linear automorphism,
- $\text{Fil}^\bullet$  is a exhausting, decreasing filtration of  $D_K := D \otimes_{K_0} K$  by  $R \otimes_{\mathbb{Q}_p} K$ -modules which are locally direct summands and satisfy  $\text{Fil}^0 = D_K$ ,  $\text{Fil}^{h+1} = (0)$ .

We can see that  $\mathcal{D}_{d,h,K}$  is a fpqc stack over  $\mathbb{Q}_p$ . Locally we can choose a basis of  $D$ ; this allows us to write the stack as a quotient

$$(5.27) \quad \mathcal{D}_{d,h,K} = [(\text{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \text{Gr}_{d,h,K}) /_{(\phi,\cdot)} \text{Res}_{K_0/\mathbb{Q}_p} GL_d].$$

Here  $\text{Gr}_{d,h,K}$  is the Grassmannian of filtrations as above of length  $h+1$  on a vector space of dimension  $d$  over  $K$ . (Notice here that we are not yet prescribing dimensions for the graded pieces  $\text{Fil}^i/\text{Fil}^{i+1}$ ; in particular,  $\text{Gr}_{d,h,K}$  is not necessarily connected.) The symbol  $(\phi,\cdot)$  is supposed to remind us that the action of the group  $\text{Res}_{K_0/\mathbb{Q}_p} GL_d$  on the product is by  $\phi$ -conjugation on the first factor and by translation on the second. It follows from this description that  $\mathcal{D}_{d,h,K}$  is an Artin stack, smooth of finite type over  $\mathbb{Q}_p$ .

Similarly, suppose that  $\mu : \bar{\mathbb{Q}}_p^\times \rightarrow (\text{Res}_{K/\mathbb{Q}_p} GL_d)(\bar{\mathbb{Q}}_p)$  is a coweight as in §3.d before. Assume that  $\mu$  is defined over the reflex field  $E$  and, for simplicity, assume  $h_- = 0$  so that  $h = h_+$ . We define the stack in groupoids  $\mathcal{D}_{\mu,K}$  over schemes over  $E$  which is described as follows: If  $R$  is an  $E$ -algebra, then the objects of  $\mathcal{D}_{\mu,K}(R)$  are triples  $(D, \Phi, \text{Fil}^\bullet)$  that correspond to objects of  $\mathcal{D}_{d,h,K}(R)$  as above with the additional property

- The filtration  $\text{Fil}^\bullet$  is of type  $\mu$  in the sense that the base change of the graded piece  $\text{Fil}^j/\text{Fil}^{j+1}$  under  $\text{id} \otimes \psi : R \otimes_{\mathbb{Q}_p} K \rightarrow R \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$  has rank equal to  $\dim(V_j^\psi)$  for each  $\psi$  and each  $j$  (see (3.23)).

Once again  $\mathcal{D}_{\mu,K}$  is a fpqc stack over  $E$  which is an Artin stack, smooth of finite type over  $E$ . We can write

$$(5.28) \quad \mathcal{D}_{\mu,K} = [(\text{Res}_{K_0/\mathbb{Q}_p} GL_d)_E \times_E \text{Gr}_{\mu,K} /_{(\phi,\cdot)} (\text{Res}_{K_0/\mathbb{Q}_p} GL_d)_E].$$

Here  $\text{Gr}_{\mu,K}$  is the Grassmannian of filtrations as above of type  $\mu$  on  $(E \otimes_{\mathbb{Q}_p} K)^d$ .

**Remark 5.1.** We can also consider the following “rigid” variants:  $\mathfrak{D}_{d,h,K} = \mathcal{D}_{d,h,K}^{\text{rig}}$  is the category fibered in groupoids over the category of rigid spaces over  $\mathbb{Q}_p$  which is defined as follows. If  $\mathfrak{X}$  is a rigid space, then  $\mathfrak{D}_{d,h,K}(\mathfrak{X})$  is the groupoid of pairs  $(D, \Phi, \text{Fil}^\bullet)$  with  $D$  a coherent sheaf of  $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K_0$ -modules over  $\mathfrak{X}$  which is locally free of rank  $d$ ,  $\Phi$  an  $1 \otimes_{\mathbb{Q}_p} \phi$ -linear isomorphism of  $D$ , and  $\text{Fil}^\bullet$  a filtration of  $D \otimes_{K_0} K$  (of length  $h$ , as above) by coherent  $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K$ -sheaves over  $\mathfrak{X}$  with locally free graded pieces. Here we are implicitly using that descent of coherent modules under fpqc morphisms of rigid spaces is effective, cf. [BG]. Similarly, we can define  $\mathfrak{D}_{\mu,K}$  etc.

5.a.1. Now suppose that  $\mathcal{A}$  is a  $p$ -adic ring. Set  $A = \mathcal{A}[1/p]$ . Let  $(\mathfrak{M}, \Phi)$  a  $\mathcal{A}_W[[u]]$ - $\Phi$ -module which corresponds to an object of  $\widehat{\mathcal{C}}_{d,h,K}(\mathcal{A})$ . Consider

$$D = (\mathfrak{M}/u\mathfrak{M})[1/p]$$

with the  $1 \otimes \phi$ -endomorphism given by  $\Phi \bmod u$ ; we can easily see that this endomorphism is bijective and that  $D$  is an  $A \otimes_{\mathbb{Q}_p} K_0$ -module which is projective of rank  $d$ .

Similarly to [Ki2], [Ki3], we set

$$\mathcal{O}_A := \lim_{\leftarrow n} (\mathcal{A}_W[[u, u^n/p]][1/p]) \subset A_W[[u]] = (A \otimes_{\mathbb{Q}_p} K_0)[[u]];$$

in particular

$$\mathcal{O} = \mathcal{O}_{\mathbb{Q}_p} := \lim_{\leftarrow n} (W[[u, u^n/p]][1/p]) \subset K_0[[u]]$$

is the ring of rigid analytic functions on the open disk  $\mathbb{U}$  of radius 1 over  $K_0$ . (The inverse limit is under the maps given by  $u^{n'}/p \mapsto u^{n'-n} \cdot (u^n/p)$ .)

We have inclusions  $W[[u]][1/p] \hookrightarrow \mathcal{O}$ ,  $\mathcal{A}_W[[u]][1/p] \hookrightarrow \mathcal{O}_A$ . The endomorphism  $\phi$  has a unique continuous extension to  $\mathcal{O}$  and  $\mathcal{O}_A$ . Set  $\mathcal{M} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$ ; the  $\Phi$ -structure on  $\mathfrak{M}$  induces

$$\Phi : \phi^*(\mathcal{M}) \rightarrow \mathcal{M}.$$

This map is injective and we have  $E(u)^h \mathcal{M} \subset \Phi(\phi^*(\mathcal{M})) \subset \mathcal{M}$ , and  $D = \mathcal{M}/u\mathcal{M}$ . Set

$$\lambda = \prod_{n=0}^{\infty} \phi^n(E(u)/E(0)) \in \mathcal{O}.$$

For each  $m$ , let  $r(m)$  be the smallest integer such that  $em < p^{r(m)}$  and consider

$$\mathcal{O}_{\mathcal{A},e,m} := \mathcal{A}_W[[u, u^{p^{r(m)}}/p^m]][1/p].$$

There is a ring homomorphism  $\mathcal{O}_A \rightarrow \mathcal{O}_{\mathcal{A},e,m}$ . Since  $|\pi| = p^{-1/e}$ , we can see that  $u \mapsto \pi$  gives  $\mathcal{O}_{\mathcal{A},e,m} \rightarrow A \otimes_{\mathbb{Q}_p} K$  which induces an isomorphism

$$(5.29) \quad \mathcal{O}_{\mathcal{A},e,m}/(E(u)) \xrightarrow{\sim} A \otimes_{\mathbb{Q}_p} K.$$

Recall  $\mathcal{M} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$ . As in [Ki3] Lemma (2.2) we see that there is a unique  $\phi$ -compatible  $A_W$ -linear map

$$\xi : D \rightarrow \mathcal{M}$$

with the following properties:

- 1) The reduction modulo  $u$  of  $\xi$  is the identity.
- 2) The induced map  $\xi : D \otimes_{A \otimes_{\mathbb{Q}_p} K_0} \mathcal{O}_A \rightarrow \mathcal{M}$  is injective and has cokernel killed by  $\lambda^h$ .
- 3) For any sufficiently large  $m$ , the induced map

$$\xi \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} : D \otimes_{A \otimes_{\mathbb{Q}_p} K_0} \mathcal{O}_{\mathcal{A},e,m} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_{\mathcal{A},e,m}$$

is injective and its image is equal to that of the map

$$\phi^*(\mathcal{M}) \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m}$$

induced by  $\Phi$ .

Indeed, the construction of  $\xi : D \rightarrow \mathcal{M}$  in [Ki3] works verbatim when  $\mathfrak{M}$  is  $\mathcal{A}_W[[u]]$ -free. (The assumptions that  $\mathcal{A}$  is complete, local and Noetherian are not needed for our version of the construction. We can choose  $\mathfrak{M}$  and  $\mathfrak{M}/u\mathfrak{M}$  to play the roles of the modules denoted by  $\mathfrak{M}_A^\circ$  and  $D_A^\circ$  in loc. cit.) The general case is obtained by gluing, using the uniqueness of  $\xi$  in the free case. To check claims (2) and (3) we can argue as in [Ki3]. (Note that loc. cit. Lemma (2.2.1) is also valid with essentially the same proof even when  $\mathcal{A}$  is not Noetherian.)

As a result of (2) and (3), and by using (5.29) to reduce modulo  $(E(u))$ , we obtain an isomorphism

$$(5.30) \quad D \otimes_{K_0} K \xrightarrow{\sim} \phi^*(\mathcal{M}) \otimes_{\mathcal{O}_A} (A \otimes_{\mathbb{Q}_p} K) \simeq \Phi(\phi^*\mathfrak{M}[1/p]) \otimes_{\mathcal{A}_W[[u]][1/p]} (A \otimes_{\mathbb{Q}_p} K).$$

(In the last tensor product, we use  $\mathcal{A}_W[[u]][1/p] \rightarrow \mathcal{A}_W[[u]][1/p]/(E(u)) \xrightarrow{\sim} A \otimes_{\mathbb{Q}_p} K$ .) We can see that the isomorphism (5.30) is independent of the choice of  $m$ .

5.a.2. Recall that  $\mathbb{U}$  denotes the rigid open unit disk over  $K_0$ . If  $I$  is a subinterval of  $[0, 1]$ , we set  $\mathbb{U}(I)$  for the admissible open subspace of points with absolute value in  $I$ . Set  $\mathcal{O}_I = \Gamma(\mathbb{U}(I), \mathcal{O}_{\mathbb{U}(I)})$  so that  $\mathcal{O} = \mathcal{O}_{[0,1]}$ . We denote by  $\phi : \mathbb{U} \rightarrow \mathbb{U}$  the “Frobenius” morphism which corresponds to  $\phi : \mathcal{O} \rightarrow \mathcal{O}$  as before.

We can consider the category  $\mathfrak{C}_{d,h,K}$  fibered in groupoids over  $\mathbb{Q}_p$ -rigid spaces which is defined as follows. Let  $\mathfrak{X}$  be a rigid space over  $\mathbb{Q}_p$  and consider  $\mathfrak{X} \times \mathbb{U}$  with the partial Frobenius  $\phi := \text{id} \times \phi : \mathfrak{X} \times \mathbb{U} \rightarrow \mathfrak{X} \times \mathbb{U}$ . Then, by definition,  $\mathfrak{C}_{d,h,K}(\mathfrak{X})$  is the groupoid of pairs  $(\mathcal{M}, \Phi)$  where  $\mathcal{M}$  is a coherent sheaf over  $\mathfrak{X} \times \mathbb{U}$  which is locally on  $\mathfrak{X}$  free of rank  $d$ , and  $\Phi : \phi^*\mathcal{M} \rightarrow \mathcal{M}$  an injective homomorphism with cokernel annihilated by  $E(u)^h$ .

Denote by  $i : \mathfrak{X}_{K_0} \hookrightarrow \mathfrak{X} \times \mathbb{U}$  the inclusion  $i(x) = (x, 0)$  and by  $p : \mathfrak{X} \times \mathbb{U} \rightarrow \mathfrak{X}_{K_0}$  the projection. If  $(\mathcal{M}, \Phi)$  is an object of  $\mathfrak{C}_{d,K}(\mathfrak{X})$  we set  $D = i^*\mathcal{M}$ . This is a coherent sheaf on  $\mathfrak{X}_{K_0}$  which is locally free of rank  $d$ ; the morphism  $\Phi : \phi^*\mathcal{M} \rightarrow \mathcal{M}$  induces a  $\phi$ -linear isomorphism  $\Phi : D \rightarrow D$ .

**Proposition 5.2.** *There is a (unique)  $\Phi$ -compatible morphism of sheaves of  $\mathcal{O}_{\mathfrak{X}_{K_0}}$ -modules  $\xi : D \rightarrow p_*(\mathcal{M})$  such that*

- 1)  $i^*\xi$  is the identity,
- 2) the induced morphism  $p^*\xi : p^*D \rightarrow \mathcal{M}$  is injective and has cokernel annihilated by  $\lambda^h$ ,
- 3) If  $r \in (|\pi|, |\pi|^{1/p})$ , then the image of the restriction  $p^*\xi_{[0,r)}$  to  $\mathfrak{X} \times \mathbb{U}[0, r)$  coincides with the image of  $\Phi_{[0,r)} : \phi^*\mathcal{M}_{[0,r)} \rightarrow \mathcal{M}_{[0,r)}$ .

*Proof.* When  $\mathfrak{X} = \text{Sp}(\mathbb{Q}_p)$  is a point, this is [Ki2] Lemma 1.2.6. Note that there is at most one  $\Phi$ -compatible  $\xi : D \rightarrow p^*(\mathcal{M})$  that satisfies property (1). Indeed, if  $\xi, \xi'$  are two such morphisms we have  $\text{Im}(\xi - \xi') \subset u \cdot p^*(\mathcal{M})$ . The  $\Phi$ -compatibility gives  $\Phi \cdot (\xi - \xi') = (\xi - \xi') \cdot \Phi$ . Hence, since  $\Phi : D \rightarrow D$  is an isomorphism, we obtain inductively  $\text{Im}(\xi - \xi') \subset u^s \cdot p^*(\mathcal{M})$  for all  $s \geq 0$ . This implies that  $\xi = \xi'$ . To show the existence of  $\xi$  we suppose first that  $A$  is a Tate  $\mathbb{Q}_p$ -algebra and that  $\mathfrak{X} = \text{Sp}(A)$  is the corresponding affinoid rigid space. Then  $\mathcal{O}_A$  is the ring of rigid analytic functions  $\mathcal{O}_{\text{Sp}(A) \times \mathbb{U}}$  on the product  $\text{Sp}(A) \times \mathbb{U}$  and  $\mathcal{M}$  is given by an  $\mathcal{O}_A$ -module as in the previous paragraph. There is a  $p$ -adic ring  $\mathcal{A}$  which is topologically of finite presentation (tfp) over  $\mathbb{Z}_p$  and  $p$ -torsion free such that  $A = \mathcal{A}[1/p]$ . The arguments of [Ki2] Lemma 1.2.6, [Ki3] Lemma (2.2) (see also the previous paragraph) extend to this case to construct  $\xi$  that satisfies all the required properties. The result in the case of a general rigid space  $\mathfrak{X}$  follows by the affinoid case above by gluing using the uniqueness of  $\xi$ .  $\square$

Suppose now that  $r$  is in  $(|\pi|, |\pi|^{1/p})$ . Then  $\mathcal{O}_{\mathfrak{X} \times \mathbb{U}[0,r]} / (E(u)) \simeq \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K$ . As a result of (2) and (3), we obtain an isomorphism

$$(5.31) \quad p^* D_{[0,r)} \xrightarrow{\sim} \Phi(\phi^* \mathcal{M})_{[0,r)}$$

which by reducing modulo  $(E(u))$  gives an isomorphism

$$(5.32) \quad D \otimes_{K_0} K \xrightarrow{\sim} \Phi(\phi^* \mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X} \times \mathbb{U}[0,r)}} (\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K)$$

of coherent  $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K$ -sheaves over  $\mathfrak{X}$ .

5.b. In what follows, we will assume that  $h = 1$ .

Let  $\mathcal{A}$  be a  $p$ -adic ring and set  $A = \mathcal{A}[1/p]$ . Suppose that  $(\mathfrak{M}, \Phi)$  is an  $\mathcal{A}_W[[u]]$ - $\Phi$ -module which corresponds to an object of  $\widehat{\mathcal{C}}_{d,K}(\mathcal{A})$ . We will define an object  $D(\mathfrak{M}, \Phi)$  of  $\mathcal{D}_{d,K}(A)$  by following the construction of [Ki2], [Ki3]. Let  $D = (\mathfrak{M}/u\mathfrak{M})[1/p]$  with its  $\Phi$ -structure be as in §5.a.1. It remains to define the filtration  $\text{Fil}^\bullet$  on  $D \otimes_{K_0} K$ . Since  $h = 1$ , we have

$$E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}.$$

The module  $\Phi(\phi^*\mathfrak{M}[1/p])$  is filtered

$$E(u)\Phi(\phi^*\mathfrak{M}[1/p]) \subset E(u)\mathfrak{M}[1/p] \subset \Phi(\phi^*\mathfrak{M}[1/p]).$$

Hence, we can filter the  $A \otimes_{\mathbb{Q}_p} K$ -module  $D \otimes_{K_0} K \simeq \Phi(\phi^*\mathfrak{M}[1/p]) \otimes_{\mathcal{A}_W[[u]][1/p]} (A \otimes_{\mathbb{Q}_p} K)$  via (5.30) by taking the image of this filtration, i.e we set

$$\begin{aligned} \text{Fil}^2 &= (0), \\ \text{Fil}^1 &= E(u)\mathfrak{M}[1/p] \text{ mod } E(u)\Phi(\phi^*\mathfrak{M}[1/p]), \\ \text{Fil}^0 &= D \otimes_{K_0} K. \end{aligned}$$

Since  $E(u)$  is not a zero divisor in  $\mathcal{A}_W[[u]][1/p]$ , we can see (cf. [Ki3] 2.6.1 (1)) that the quotient  $\mathfrak{M}[1/p]/\Phi(\phi^*\mathfrak{M}[1/p])$  is a finitely generated projective  $A \otimes_{\mathbb{Q}_p} K$ -module. We conclude that  $\text{Fil}^1$  is a finitely generated projective  $A \otimes_{\mathbb{Q}_p} K$ -module which is locally a direct summand of  $D \otimes_{K_0} K$ . Hence,  $(D, \Phi, \text{Fil}^\bullet)$  gives an object of  $\mathcal{D}_{d,K}(A)$  which we will denote by  $D(\mathfrak{M}, \Phi)$ . This gives a functor of groupoids,

$$\underline{D}(\mathcal{A}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \rightarrow \mathcal{D}_{d,K}(A).$$

5.b.1. Similarly, if  $(\mathcal{M}, \Phi)$  is an object of  $\mathfrak{C}_{d,K}(\mathfrak{X})$  for a rigid space  $\mathfrak{X}$ , we consider  $D = i^*\mathcal{M}$  with its  $\phi$ -linear isomorphism  $\Phi : D \rightarrow D$  as above. We also have

$$E(u)\Phi(\phi^*\mathcal{M}) \subset E(u)\mathcal{M} \subset \Phi(\phi^*\mathcal{M})$$

(A filtration of coherent sheaves over  $\mathfrak{X} \times \mathbb{U}$ .) As above, we can use this and (5.32) to produce a filtration

$$(0) = \text{Fil}^2 \subset \text{Fil}^1 \subset \text{Fil}^0 = D \otimes_{K_0} K$$

of the coherent sheaf  $D \otimes_{K_0} K$  over  $\mathfrak{X}_{K_0}$ . We can see that the triple  $(D, \Phi, \text{Fil}^\bullet)$  gives an object of  $\mathfrak{D}_{d,K}(\mathfrak{X})$ . Since  $(\mathcal{M}, \Phi) \mapsto (D, \Phi, \text{Fil}^\bullet)$  is functorial, this defines a functor of

groupoids,  $\mathfrak{C}_{d,K}(\mathfrak{X}) \rightarrow \mathfrak{D}_{d,K}(\mathfrak{X})$ . This functor is compatible with descent, hence we obtain a morphism of stacks over the category of rigid spaces,

$$(5.33) \quad \underline{D} : \mathfrak{C}_{d,K} \rightarrow \mathfrak{D}_{d,K}.$$

Similarly, if  $\mu$  is a minuscule cocharacter with  $h_- = 0$ ,  $h_+ = 1$  and with reflex field  $E$  as in Remark 3.3, we can define the category  $\mathfrak{C}_{\mu,K}$  fibered in groupoids over  $E$ -rigid spaces by requiring the cokernels  $\mathcal{M}/\Phi(\phi^*\mathcal{M})$  to have a filtration “of type  $\mu$ ”. In this case, the morphism  $\underline{D}$  sends  $\mathfrak{C}_{\mu,K}$  to  $\mathfrak{D}_{\mu,K}$ .

5.b.2. It follows from [Ki2] Theorem (1.2.15) that the functor  $\underline{D}(\mathrm{Sp}(L))$  gives an equivalence  $\mathfrak{C}_{d,K}(\mathrm{Sp}(L)) \xrightarrow{\sim} \mathfrak{D}_{d,K}(\mathrm{Sp}(L))$  for any finite extension  $L/\mathbb{Q}_p$ . To briefly explain the construction of the inverse functor we need some notation: Denote by  $\sigma$  the isomorphism  $\mathcal{O} \rightarrow \mathcal{O}$  given by applying Frobenius (only) to the coefficients of the power series. Denote by  $x_n$  the point of  $\mathbb{U}$  that corresponds to the irreducible polynomial  $E(u^{p^n})$  and let  $\hat{\mathcal{O}}_{\mathbb{U},x_n}$  the complete local ring of  $\mathbb{U}$  at  $x_n$ . Notice that the function  $\sigma^{-n}(\lambda) \in \mathcal{O}$  has a simple zero at  $x_n$ . Now consider the composite map

$$\mathcal{O} \otimes_{K_0} D \xrightarrow{\sigma^{-n} \otimes \Phi^{-n}} \mathcal{O} \otimes_{K_0} D \rightarrow \hat{\mathcal{O}}_{\mathbb{U},x_n} \otimes_{K_0} D = \hat{\mathcal{O}}_{\mathbb{U},x_n} \otimes_K D_K,$$

where in the first arrow  $\Phi^{-n} : D \rightarrow D$  makes sense since  $\Phi = \Phi_D$  is bijective. By the above, this induces a map

$$i_n : \mathcal{O}[\lambda^{-1}] \otimes_{K_0} D \rightarrow \hat{\mathcal{O}}_{\mathbb{U},x_n}[(u - x_n)^{-1}] \otimes_K D_K.$$

Now suppose we are given an object  $(D, \Phi, \mathrm{Fil}^\bullet)$  over  $L$ . Kisin constructs a  $\Phi$ -module  $(\mathcal{M}, \Phi) = \mathcal{M}(D, \Phi, \mathrm{Fil}^\bullet)$  over  $\mathcal{O}_L = \mathcal{O}_{\mathrm{Sp}(L) \times \mathbb{U}}$  by taking

$$(5.34) \quad \mathcal{M} = \bigcap_{n \geq 0} i_n^{-1}(\mathrm{Fil}^1 \otimes_K (u - x_n)^{-1} \hat{\mathcal{O}}_{\mathbb{U},x_n} + D_K \otimes_K \hat{\mathcal{O}}_{\mathbb{U},x_n})$$

and setting  $\Phi : \phi^*\mathcal{M} \rightarrow \mathcal{M}$  to be the restriction of

$$1 \otimes \Phi_D : \phi^*(\mathcal{O}[\lambda^{-1}] \otimes_{K_0} D) \rightarrow \mathcal{O}[\lambda^{-1}] \otimes_{K_0} D.$$

Note that by definition, we have

$$(5.35) \quad \mathcal{O} \otimes_{K_0} D \subset \mathcal{M} \subset \lambda^{-1} \mathcal{O} \otimes_{K_0} D.$$

Observe that by its construction,  $\mathcal{M}$  is a closed  $\mathcal{O}$ -submodule of  $\lambda^{-1} \mathcal{O} \otimes_{K_0} D$  and so by [Ki2] Lemma 1.1.4,  $\mathcal{M}$  is finite free over  $\mathcal{O}$ .

5.b.3. This construction extends to the case that  $L$  is a complete rank-1 valued field: Let  $R^\circ$  be a  $p$ -adic valuation ring of rank 1 with  $L = R^\circ[1/p]$ . Then  $\mathcal{O}_L = \mathcal{O}_{[0,1]}$  is the ring of rigid functions on the open unit disk over  $L \otimes_{\mathbb{Q}_p} K_0$ . For simplicity, set  $L_{K_0} = L \otimes_{\mathbb{Q}_p} K_0$ ,  $L_K = L \otimes_{\mathbb{Q}_p} K$ . Consider a triple  $(D, \Phi, \text{Fil}^\bullet)$  as above. We can construct a  $\Phi$ -module  $\mathcal{M}$  over  $\mathcal{O}_{[0,1]}$  by (5.34) as above that satisfies

$$(5.36) \quad \mathcal{O}_{[0,1]} \otimes_{L_{K_0}} D \subset \mathcal{M} \subset \lambda^{-1} \mathcal{O}_{[0,1]} \otimes_{L_{K_0}} D .$$

By [Gr], V, Rem. 3°, p. 87,  $\mathcal{O}_{[0,1]}$  is a product of Prüfer domains. We can see that when an integral power of  $r$  is in the set  $|L|$  the restriction  $\mathcal{M}|_{[0,r]}$  of  $\mathcal{M}$  to the closed disk  $[0, r]$  is given by a finitely generated torsion free  $\mathcal{O}_{[0,r]}$ -module which is free (since  $\mathcal{O}_{[0,r]}$  is a product of p.i.ds). Using [Gr], V, Thm. 1, p. 83, we can see that  $\mathcal{M}$  is a projective finitely generated  $\mathcal{O}_{[0,1]}$ -module. As such it is a direct sum of a free module with a projective module  $\mathcal{L}$  of rank 1 and  $\mathcal{L} \simeq \det(\mathcal{M})$ . We can see that  $\det(\mathcal{M}) = \lambda^{-a} \mathcal{O}_{[0,1]}$  with  $a = \dim_{L_K}(\text{Fil}^1)$ ; therefore  $\mathcal{L}$  and hence  $\mathcal{M}$  is finite free over  $\mathcal{O}_{[0,1]}$ .

5.b.4. The constructions of the two previous paragraphs are compatible in the following sense: Suppose that the  $p$ -adic ring  $\mathcal{A}$  is topologically of finite presentation over  $\mathbb{Z}_p$ . Then  $A = \mathcal{A}[1/p]$  is a Tate algebra and we can consider the affinoid rigid space  $\text{Sp}(A)$ . Recall that  $\text{Sp}(A)$  is the “generic fiber”  $\text{Spf}(\mathcal{A})^{\text{rig}}$  of  $\text{Spf}(\mathcal{A})$  in the sense of Raynaud. An object  $(\mathfrak{M}, \Phi)$  of  $\widehat{\mathcal{C}}_{d,K}(\mathcal{A})$  gives an object  $(\mathcal{M}, \Phi)$  of  $\mathfrak{C}_{d,K}(\text{Sp}(A))$  by taking  $\mathcal{M}$  to be the coherent sheaf with global sections  $\mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$ . This gives a functor  $\widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \rightarrow \mathfrak{C}_{d,K}(\text{Sp}(A))$ . The diagram

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) & \longrightarrow & \mathfrak{C}_{d,K}(\text{Sp}(A)) \\ D(\mathcal{A}) \downarrow & & \downarrow D(\text{Sp}(A)) \\ \mathcal{D}_{d,K}(A) & \longrightarrow & \mathfrak{D}_{d,K}(\text{Sp}(A)) , \end{array}$$

commutes up to natural equivalence. (Here the lower horizontal arrow is given by sending the  $A$ -module  $D$  to the corresponding coherent sheaf over  $\text{Sp}(A)$ .) The diagonal arrow

$$\Pi(\mathcal{A}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \longrightarrow \mathfrak{D}_{d,K}(\text{Sp}(A))$$

(obtained as the composition of the top followed by the right downward arrow) is by definition, the period functor for  $\mathcal{A}$ . It globalizes as follows.

Suppose that  $\mathcal{X}$  is an adic formal scheme which is locally of finite type over  $\mathbb{Z}_p$  (hence  $p\mathcal{O}_{\mathcal{X}}$  is an ideal of definition), and denote by  $\mathfrak{X} = \mathcal{X}^{\text{rig}}$  the corresponding rigid space given by its generic fiber, comp. [RZ], Prop. 5.3. The construction  $(\mathfrak{M}, \Phi) \mapsto (\mathcal{M}, \Phi)$  above generalizes to give a functor  $\widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \rightarrow \mathfrak{C}_{d,K}(\mathcal{X}^{\text{rig}})$ . Its composition with the functor  $\omega(\mathcal{X}^{\text{rig}}) : \mathfrak{C}_{d,K}(\mathcal{X}^{\text{rig}}) \rightarrow \mathfrak{D}_{d,K}(\mathcal{X}^{\text{rig}})$  above gives the *period functor*

$$(5.37) \quad \Pi(\mathcal{X}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \longrightarrow \mathfrak{D}_{d,K}(\mathcal{X}^{\text{rig}}) .$$

It is localizing in the following sense. Let  $\mathcal{X} = \bigcup_i \mathcal{U}_i$  be an open covering of the formal scheme  $\mathcal{X}$ . This induces an admissible open covering of the associated rigid-analytic spaces,

$$\mathcal{X}^{\text{rig}} = \bigcup_i \mathcal{U}_i^{\text{rig}},$$

comp. [RZ], Prop. 5.3. Then the corresponding diagram of 2-cartesian rows, with vertical arrows the period morphisms, is commutative,

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \rightarrow & \prod_i \widehat{\mathcal{C}}_{d,K}(\mathcal{U}_i) & \rightrightarrows \prod_{i,j} \widehat{\mathcal{C}}_{d,K}(\mathcal{U}_i \cap \mathcal{U}_j) \\ \downarrow & \downarrow & \downarrow \\ \mathfrak{D}_{d,K}(\mathcal{X}^{\text{rig}}) \rightarrow & \prod_i \mathfrak{D}_{d,K}(\mathcal{U}_i^{\text{rig}}) & \rightrightarrows \prod_{i,j} \mathfrak{D}_{d,K}(\mathcal{U}_i^{\text{rig}} \cap \mathcal{U}_j^{\text{rig}}). \end{array}$$

5.b.5. Let us now assume that  $R^\circ$  is a complete rank one valuation ring with residue field equal to  $\bar{\mathbb{F}}_p$  and set  $L = R^\circ[1/p]$  for the corresponding complete rank-1 valued field. Recall  $\mathcal{O}_L = \mathcal{O}_{[0,1)}$  is the ring of rigid functions on the open unit disk over  $L_{K_0}$ . Denote by  $\mathcal{O}_L^R$  the corresponding Robba ring

$$\mathcal{O}_L^R := \lim_{r \rightarrow 1^-} \mathcal{O}_{(r,1)}.$$

This can be identified with the set of Laurent power series  $\sum_{n \in \mathbb{Z}} a_n u^n$  with coefficients in  $L_{K_0}$  that converge in some open annulus  $r < |u| < 1$ . The ring  $\mathcal{O}_L^R$  is equipped with a Frobenius endomorphism  $\phi : \mathcal{O}_L^R \rightarrow \mathcal{O}_L^R$  which restricts to  $\phi : \mathcal{O}_L \rightarrow \mathcal{O}_L$ . Let  $\mathcal{O}_L^{\text{int}}$  the subring of  $\mathcal{O}_L^R$  consisting of those Laurent power series  $\sum_{n \in \mathbb{Z}} a_n u^n$  with  $a_n \in R^\circ \otimes_{\mathbb{Z}_p} W$ , for all  $n \in \mathbb{Z}$ . By [Ke2] Prop. 3.5.5,  $\mathcal{O}_L^{\text{int}}$  is a henselian local ring with maximal ideal given by the set of series with  $|a_n| < 1$ , for all  $n \in \mathbb{Z}$ , and residue field  $\bar{\mathbb{F}}_p((u))$ . Notice that  $E(u)$  is a unit in  $\mathcal{O}_L^{\text{int}}$  and  $\phi$  preserves  $\mathcal{O}_L^{\text{int}}$ .

Suppose now that  $\mathcal{N}$  is a finite free rank  $d$   $\Phi$ -module over  $\mathcal{O}_L^R$  with  $\Phi : \phi^* \mathcal{N} \xrightarrow{\sim} \mathcal{N}$  an isomorphism. We will say that  $(\mathcal{N}, \Phi)$  is *purely of slope zero* if there is a finite free rank  $d$   $\mathcal{O}_L^{\text{int}}$ -submodule  $\mathcal{N}^{\text{int}} \subset \mathcal{N}$  such that:

- i)  $\mathcal{N}^{\text{int}} \otimes_{\mathcal{O}_L^{\text{int}}} \mathcal{O}_L^R = \mathcal{N}$ , and
- ii)  $\Phi|_{\mathcal{N}^{\text{int}}}$  induces an isomorphism  $\phi^* \mathcal{N}^{\text{int}} \xrightarrow{\sim} \mathcal{N}^{\text{int}}$ .

When  $L/\mathbb{Q}_p^{\text{unr}}$  is finite, this is equivalent to asking that  $(\mathcal{N}, \Phi)$  is purely of slope zero in the sense of Kedlaya [Ke1].

As in the previous paragraph, we have the period functor

$$\Pi(R^\circ) : \widehat{\mathcal{C}}_{d,K}(R^\circ) \rightarrow \mathcal{D}_{d,K}(L) \simeq \mathfrak{D}_{d,K}(\text{Sp}(L)).$$

By 5.b.3 we can associate to an object  $(D, \Phi, \text{Fil}^\bullet)$  of  $\mathcal{D}_{d,K}(L)$  a  $\Phi$ -module  $(\mathcal{M}, \Phi) = \mathcal{M}(D, \Phi, \text{Fil}^\bullet)$  over  $\mathcal{O}_L$ .

**Conjecture 5.3.** (i) *The object  $(D, \Phi, \text{Fil}^\bullet)$  of  $\mathfrak{D}_{d,K}(\text{Sp}(L))$  is in the image of the period functor  $\Pi(R^\circ)$ , i.e is of the form  $\Pi(R^\circ)(\mathfrak{M}, \Phi)$  for some  $(\mathfrak{M}, \Phi) \in \widehat{\mathcal{C}}_{d,K}(R^\circ)$ , if and only if  $\mathcal{M}(D, \Phi, \text{Fil}^\bullet) \otimes_{\mathcal{O}_L} \mathcal{O}_L^R$  is purely of slope zero.*

(ii) There exists an  $\mathbb{Q}_p$ -analytic subspace (in the sense of Berkovich)

$$(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K})^{\mathrm{an}} \subset \mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K}$$

invariant under  $\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d$  such that the fiber over  $L$  of the stack quotient

$$[(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K})^{\mathrm{an}} /_{(\phi,\cdot)} \mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d]$$

parametrizes the image points of  $\Pi$  over  $L$ .

There is an obvious variant of this conjecture involving a minuscule cocharacter  $\mu$  with  $h_- = 0, h_+ = 1$ .

## 6. KISIN VARIETIES AND BRUHAT-TITS BUILDINGS

6.a. We return to the set-up and notations of §2.c.1. Let  $\mathbb{F}$  be a finite extension of  $\mathbb{F}_p$ .

6.a.1. For simplicity, set  $L = \mathbb{F} \otimes k$ . Recall  $G = \mathrm{Res}_{W/\mathbb{Z}_p} GL_d$ . Suppose now that  $A \in G(\mathbb{F}((u))) = GL_d(L((u)))$  and consider the corresponding  $L((u))$ - $\Phi$ -module  $M_A = (L((u))^d, A \cdot \phi)$  which gives an object of  $\mathcal{R}(\mathbb{F})$ . Choices  $A, A'$  that are  $\phi$ -conjugate, i.e  $A' = g^{-1} \cdot A \cdot \phi(g)$  with  $g \in GL_d(L((u)))$ , give isomorphic modules. By the above, the fiber product  $\{M_A\} \times_{\mathcal{R}} \mathcal{C}_\nu$  is represented over  $\mathbb{F}$  by a projective subscheme of the affine Grassmannian  $\mathcal{F}_G$  of  $L[[u]]$ -lattices in  $L((u))^d$ . We denote this subscheme by  $\mathcal{C}_\nu(A)$ . Similarly, we can consider the fiber product  $\{M_A\} \times_{\mathcal{R}} \mathcal{C}_\mu^0$  which is a locally closed subscheme of  $\mathcal{C}_\nu(A)$ ; we denote this subscheme by  $\mathcal{C}_\nu^0(A)$ . This can be thought of as an inseparable analogue of an affine Deligne-Lusztig variety. We call  $\mathcal{C}_\nu^0(A)$  the *Kisin variety* associated to  $(G, A, \nu)$ , and  $\mathcal{C}_\nu(A)$  the corresponding *closed Kisin variety*.

Concretely, for every finite extension  $\mathbb{F}'$  of  $\mathbb{F}$ , the  $\mathbb{F}'$ -points of the Kisin variety  $\mathcal{C}_\nu^0(A)$  are given by

$$\mathcal{C}_\nu^0(A)(\mathbb{F}') = \{g \cdot (\mathbb{F}' \otimes k)[[u]]^d \mid g^{-1} \cdot A \cdot \phi(g) \in G(\mathbb{F}'[[u]]) \cdot u^\nu \cdot G(\mathbb{F}'[[u]])\}.$$

The points  $\mathcal{C}_\nu(A)(\mathbb{F}')$  parametrize finite flat commutative group schemes  $\mathcal{G}$  with  $\mathbb{F}'$ -action over  $\mathcal{O}_K$  which have “Hodge type  $\leq \nu$ ” and are such that the restriction of the Galois representation  $\mathrm{Gal}(\bar{K}/K) \rightarrow \mathrm{Aut}_{\mathbb{F}'}(\mathcal{G}(\bar{\mathcal{O}}_K))$  to the Galois group  $G_\infty$  corresponds to the  $\Phi$ -module given by  $A$ .

6.a.2. The above construction extends to the set-up of a general reductive group  $G = \mathrm{Res}_{W/\mathbb{Z}_p} H$  described in §2.d. If  $\nu$  is a dominant coweight of  $G$  and  $A \in LG(\mathbb{F}) = G(\mathbb{F}((u)))$ , we can define as above  $\mathcal{C}_{\nu,G}$ ,  $\mathcal{C}_{\nu,G}$  and the Kisin variety  $\mathcal{C}_{\nu,G}^0(A)$ , and closed Kisin variety  $\mathcal{C}_{\nu,G}(A)$ . However, the relation with Galois representations of  $\mathrm{Gal}(\bar{K}/K)$  or finite group schemes is not so clear in this general case.

6.b. We now explain how the Bruhat-Tits building can help to get an overview of a Kisin variety.

6.b.1. For simplicity, we assume that  $k = \mathbb{F}_p$ ,  $W = \mathbb{Z}_p$  and that  $H = G$  is a split Chevalley group over  $\mathbb{Z}_p$ . In the rest of this section, the symbol  $W$  is free again, and will be reserved for Weyl groups. Let  $T$  be a maximal split torus of  $G$ . We will identify the cocharacter groups  $X_* = X_*(T_{\mathbb{Q}_p}) = X_*(T_{\mathbb{F}_p}) = X_*(T_{\mathbb{F}_p((u))})$ . Suppose that  $C$  is a choice of a positive closed Weyl chamber in the vector space

$$V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let  $\mathcal{B} = \mathcal{B}(\mathbb{F}((u)))$  be the Bruhat-Tits building of  $G$  over  $\mathbb{F}((u))$ . This is a metric space with equivariant distance function  $d : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$ . We have the “refined” Weyl distance function  $\delta : \mathcal{B} \times \mathcal{B} \rightarrow C$  which is defined as follows, cf. [KLM], section 5.1: Let  $x, y \in \mathcal{B}$  and suppose that  $\mathcal{A}$  is an apartment that contains both  $x$  and  $y$ . Let  $\delta_{\mathcal{A}}(x, y)$  be the unique representative in  $C$  of the vector  $y - x \in V$  and set  $\delta(x, y) = \delta_{\mathcal{A}}(x, y)$ . (This is independent of the choice of apartment  $\mathcal{A}$ .) The function  $\delta$  is translation  $G$ -equivariant and satisfies the triangle inequality, cf. [KLM], Remark 3.33, (ii),

$$(6.38) \quad \delta(x, z) \leq \delta(x, y) + \delta(y, z)$$

for the order that extends the usual order on dominant coweights. Also,

$$(6.39) \quad \delta(x, y) = \delta(y, x)^*,$$

where  $v \mapsto v^* = w_0(-v)$  is the usual involution of  $C$  defined by the longest element  $w_0$  of the finite Weyl group  $W$ .

6.b.2. Consider now the homomorphism  $\phi : \mathbb{F}((u)) \rightarrow \mathbb{F}((u))$ , given  $\phi(a) = a$  if  $a \in \mathbb{F}$ ,  $\phi(u) = u^p$ . We will show that it induces a map  $\phi : \mathcal{B} \rightarrow \mathcal{B}$  with the following properties:

- (a) the image of any apartment under  $\phi$  is an apartment,
- (b) we have  $\phi(g) \cdot \phi(x) = \phi(g \cdot x)$  for any  $g \in G(\mathbb{F}((u)))$ ,  $x \in \mathcal{B}$ .
- (c) For  $x, y \in \mathcal{B}$ , we have

$$d(\phi(x), \phi(y)) = p \cdot d(x, y), \quad \delta(\phi(x), \phi(y)) = p \cdot \delta(x, y).$$

- (d) The map  $\phi : \mathcal{B} \rightarrow \mathcal{B}$  takes maps geodesics to geodesics; i.e., if  $[x, y] \subset \mathcal{B}$  is the geodesic in  $\mathcal{B}$  joining  $x$  and  $y$ , then the image  $\phi([x, y])$  is the geodesic  $[\phi(x), \phi(y)]$  joining  $\phi(x)$  and  $\phi(y)$ .
- (e) The map  $\phi$  has a unique fixed point, i.e., there is a unique  $y_0 \in \mathcal{B}$  such that  $\phi(y_0) = y_0$ . The point  $y_0$  is a special vertex in  $\mathcal{B}$ .

Indeed, consider the vertex  $y_0$  of  $\mathcal{B}$  which is fixed under the subgroup  $G(\mathbb{F}[[u]])$ . Let  $\mathcal{A}_0$  be the apartment in  $\mathcal{B}$  that corresponds to a constant maximal torus  $T = T_0 \otimes_{\mathbb{F}} \mathbb{F}((u))$  with  $T_0 \subset G_{\mathbb{F}}$ ; then  $y_0$  belongs to  $\mathcal{A}_0$  and this choice of base point allows us to identify the affine space  $\mathcal{A}_0$  with  $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . Scaling by  $p$  on  $V$  now gives a well-defined map  $\phi_0 : \mathcal{A}_0 \rightarrow \mathcal{A}_0$  such that  $\phi_0(y_0) = y_0$  and which satisfies  $\phi_0(n \cdot y) = \phi(n) \cdot \phi_0(y)$  for each  $y \in \mathcal{A}_0$  and  $n$  in the normalizer  $N(T) \subset G(\mathbb{F}((u)))$ . Now recall that the building  $\mathcal{B}$  can be described as the quotient of  $G(\mathbb{F}((u))) \times \mathcal{A}_0$  via the equivalence relation  $(g, x) \sim (x', g')$  if

there is  $n \in N(T)$  such that  $x' = n \cdot x$ ,  $g' = ngn^{-1}$ . Using the above we immediately see that  $\phi(x, g) = (\phi_0(x), \phi(g))$  respects the equivalence relation and gives  $\phi : \mathcal{B} \rightarrow \mathcal{B}$ ; since each apartment of  $\mathcal{B}$  is of the form  $g \cdot \mathcal{A}_0$  we see that the image of an apartment by  $\phi$  is also an apartment. The desired properties now follow easily by using the above and the fact that any two points  $x, y \in \mathcal{B}$  are contained in some apartment and that the geodesic  $[x, y]$  is the straight line segment connecting  $x$  and  $y$  in that apartment. Note that the equality  $d(\phi(x), \phi(y)) = p \cdot d(x, y)$  implies that there is at most one fixed point which then has to be the vertex  $y_0$  given above.

Note that, by construction, the group  $L^+G(\mathbb{F}) = G(\mathbb{F}[[u]])$  is the stabilizer of  $y_0$  in  $G(\mathbb{F}((u)))$ . The map

$$(6.40) \quad \iota : \mathcal{F}_G(\mathbb{F}) = G(\mathbb{F}((u)))/G(\mathbb{F}[[u]]) \hookrightarrow \mathcal{B}, \quad g \cdot G(\mathbb{F}[[u]]) \mapsto g \cdot y_0$$

allows us to identify the  $\mathbb{F}$ -valued points of the affine Grassmannian with a subset of the vertices in the building.

6.b.3. Suppose now that  $A$  is in  $G(\mathbb{F}((u)))$  and gives an object in  $\mathcal{R}_G(\mathbb{F})$ . Then we have a map  $\Phi = A \cdot \phi : \mathcal{B} \rightarrow \mathcal{B}$  which also satisfies

$$(6.41) \quad d(\Phi_A(x), \Phi_A(y)) = p \cdot d(x, y), \quad \delta(\Phi_A(x), \Phi_A(y)) = p \cdot \delta(x, y).$$

Then the  $\mathbb{F}$ -valued points of  $\mathcal{C}_\nu(A) \subset \mathcal{F}_G(\mathbb{F})$  correspond to the following subset of vertices of the building,

$$\mathcal{C}_\nu(A) = \{x \text{ vertex in } \mathcal{B} \mid x \in \text{Im } \iota, 0 \leq \delta(x, \Phi_A(x)) \leq \nu\}.$$

If  $N/\mathbb{F}((u))$  is a finite separable extension, we have an isometric embedding

$$\mathcal{B} \hookrightarrow \mathcal{B}(N).$$

We will use this to identify  $\mathcal{B}$  with a subspace of  $\mathcal{B}(N)$ . The map  $\Phi_A$  extends to a map  $\mathcal{B}(N) \rightarrow \mathcal{B}(N)$ .

**Proposition 6.1.** *There is a finite separable extension  $M/\mathbb{F}((u))$  such that the above map  $\Phi_A : \mathcal{B}(M) \rightarrow \mathcal{B}(M)$  has a fixed point. This fixed point is unique in  $\cup_{N/\mathbb{F}((u))} \mathcal{B}(N)$ .*

*Proof.* The uniqueness follows easily from (6.41). For simplicity, we will write  $\Phi$  instead of  $\Phi_A$ . Consider the “Lang isogeny”

$$G_{\mathbb{F}((u))} \rightarrow G_{\mathbb{F}((u))}; \quad g \mapsto g^{-1}\phi(g).$$

This is a finite étale surjective morphism; therefore, if  $A$  is in  $G(\mathbb{F}((u)))$ , then there is  $g \in G(M)$  for some finite Galois extension  $M/\mathbb{F}((u))$  such that  $A = g^{-1}\phi(g)$ . Consider  $x_0 = g^{-1} \cdot y_0$  which is a special vertex in  $\mathcal{B}(M)$ . We have

$$\Phi(x_0) = A \cdot \phi(g^{-1} \cdot y_0) = g^{-1} \cdot \phi(g) \cdot \phi(g)^{-1} \cdot y_0 = g^{-1} \cdot y_0 = x_0$$

and so  $x_0$  is a fixed point. □

In fact, if  $\sigma$  is an element of  $\text{Gal}(M/\mathbb{F}((u)))$ , since  $\sigma \cdot \phi = \phi \cdot \sigma$ , we can see that  $\sigma(A) = \sigma(g^{-1}\phi(g)) = \sigma(g)^{-1}\phi(\sigma(g)) = A = g^{-1}\phi(g)$ ; therefore  $\sigma(g)g^{-1} \in G(\mathbb{F})$ . Since  $g_0 \cdot y_0 = y_0$  for  $g_0 \in G(\mathbb{F})$ , the point  $x_0 = g^{-1} \cdot y_0$  depends only on  $A$  and is  $\text{Gal}(M/\mathbb{F}((u)))$ -fixed. Therefore, if  $M/\mathbb{F}((u))$  is tamely ramified, which implies  $\mathcal{B}(M)^{\text{Gal}(M/\mathbb{F}((u)))} = \mathcal{B}$ , we conclude that  $x_0$  belongs to  $\mathcal{B}$ .

6.b.4. We continue to write  $\Phi = \Phi_A$ . If  $x$  is in  $\mathcal{B}$ , we can apply the triangle inequality above to  $x$ ,  $\Phi(x)$  and  $x_0 = \Phi(x_0)$ , in two different ways. We obtain:

$$\begin{aligned} \delta(\Phi(x), x) &\leq \delta(\Phi(x), x_0) + \delta(x_0, x) = p \cdot \delta(x, x_0) + \delta(x, x_0)^*, \\ \delta(\Phi(x), x_0) &= p \cdot \delta(x, x_0) \leq \delta(\Phi(x), x) + \delta(x, x_0). \end{aligned}$$

Combining these we get

$$(6.42) \quad (p-1) \cdot \delta(x, x_0) \leq \delta(\Phi(x), x) \leq p \cdot \delta(x, x_0) + \delta(x, x_0)^*.$$

This implies that if  $h \in G(\mathbb{F}((u)))$  is such that

$$(6.43) \quad p \cdot \delta(h \cdot y_0, x_0) + \delta(h \cdot y_0, x_0)^* \leq \nu,$$

then the corresponding point  $h \cdot G(\mathbb{F}[[u]])$  in  $\mathcal{F}_G(\mathbb{F})$  belongs to  $\mathcal{C}_{\nu, G}(A)$ , which is then non-empty and is contained in the ball of radius  $\nu/(p-1)$  around  $x_0$ .

6.b.5. Suppose that  $A' = h^{-1} \cdot A \cdot \phi(h)$  with  $h \in G(\mathbb{F}((u)))$ . Then  $A' = (gh)^{-1}\phi(gh)$  and the corresponding  $\Phi_{A'}$ -fixed vertex is  $x'_0 = (gh)^{-1} \cdot y_0 = h^{-1} \cdot x_0$ . We conclude that the orbit  $G(\mathbb{F}((u))) \cdot x_0$  only depends on the  $\phi$ -conjugacy class of  $A$  in  $G(\mathbb{F}((u)))$ . By the above, if  $M/\mathbb{F}((u))$  is tamely ramified,  $x_0$  belongs to  $\mathcal{B}$ .

6.c. We continue to assume that  $k = \mathbb{F}_p$  and now take  $G = H = GL_d$ . Take  $T$  the standard maximal torus of  $GL_d$ . Then the finite Weyl group is the symmetric group  $S_d$ ,  $X_*(T)_{\mathbf{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ , and the standard choice of a positive closed Weyl chamber is

$$C = \{(v_1, \dots, v_d) \in \mathbb{R}^d \mid v_1 \geq v_2 \geq \dots \geq v_d\}.$$

The partial order on  $C$  is given by:  $(v_1, \dots, v_d) \leq (v'_1, \dots, v'_d)$  iff

$$\sum_{i=1}^r v_i \leq \sum_{i=1}^r v'_i, \text{ for } r = 1, \dots, d-1, \text{ and } v_1 + \dots + v_d = v'_1 + \dots + v'_d.$$

In this case, we will explain the construction of the fixed point in a slightly different way. Start with  $M_A = (k((u))^d, A \cdot \phi)$  and set

$$U = (k((u))^{\text{sep}} \otimes_{k((u))} M_A)^{\phi \otimes \Phi = \text{Id}} \subset k((u))^{\text{sep}} \otimes_{k((u))} M_A$$

for the  $k$ -vector space of the corresponding  $\text{Gal}(k((u))^{\text{sep}}/k((u)))$ -representation  $\rho$ . (Here  $\phi : k((u))^{\text{sep}} \rightarrow k((u))^{\text{sep}}$  denotes again the Frobenius of the separable closure.) In fact, one can see from the construction of  $\rho$  that there is a finite separable extension  $L/k((u))$  such that

$$U = (L \otimes_{k((u))} M_A)^{\phi \otimes \Phi = \text{Id}} \subset L \otimes_{k((u))} M_A = L^d$$

as  $\text{Gal}(k((u))^{\text{sep}}/k((u)))$ -modules. (Note that  $L^d$  also supports a  $\Phi$ -module structure for the extension  $\phi|_L$  of  $\phi$  to  $L$ ; this is given by  $A \cdot \phi|_L$ ) Now set  $\mathfrak{M}_0$  for the  $\mathcal{O}_L$ -submodule in  $L^d$  generated by the elements in  $U$ . Then  $\mathfrak{M}_0/u_L\mathfrak{M}_0 \simeq U$  and so  $\mathfrak{M}_0$  is an  $\mathcal{O}_L$ -lattice in  $L^d$ . Since  $\phi \otimes \Phi = A \cdot \phi|_L$  acts as identity on  $U$ , we can see that

$$(A \cdot \phi|_L)^*(\mathfrak{M}_0) = \mathfrak{M}_0.$$

The lattice  $\mathfrak{M}_0$  gives a point  $x_0$  of  $\mathcal{B}(L)$  which is fixed under the map  $\Phi$ , i.e  $\Phi(x_0) = x_0$ .

6.d. In this paragraph, we will explain the picture in the building for  $\mathbb{F} = \mathbb{F}_p$  and  $G = H = GL_2$ . Our main objective is the following. Given a dominant coweight  $\nu = (a, b)$  with  $a \geq b \geq 0$  and a matrix  $A \in GL_2(\mathbb{F}((u)))$ , describe the set of vertices in the building  $\mathcal{B}$  which correspond to  $\mathbb{F}$ -valued points in  $\mathcal{C}_\nu(A)$ , i.e, to lattices  $\mathfrak{M} \subset \mathbb{F}((u))^2$  for which  $\Phi_A(\phi^*(\mathfrak{M})) \subset \mathfrak{M}$  and such that  $\mathfrak{M}/\Phi_A(\phi^*(\mathfrak{M})) = \mathfrak{M}/\langle A \cdot \phi(\mathfrak{M}) \rangle$  has elementary divisors  $(a', b')$ ,  $a' \geq b' \geq 0$  which are smaller than  $\nu = (a, b)$ , i.e  $a' \leq a$ ,  $a' + b' = a + b$ . The corresponding set in the building is the set of vertices  $x$  such that  $0 \leq \delta(x, \Phi_A(x)) \leq \nu$ . To simplify our discussion, we will consider the projection  $\mathcal{B} \rightarrow \mathcal{T}$  where  $\mathcal{T}$  is the tree of homothety classes of lattices in  $\mathbb{F}((u))^2$  (i.e the building for  $PGL_2(\mathbb{F}((u)))$ ). Note that the Weyl chamber distance  $\delta$  on the tree  $\mathcal{T}$  coincides (up to sign) with the usual distance  $d : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$ .

We consider the sets  $\text{Vert}(\mathcal{T})_{\nu, A}$  of vertices  $x$  in the tree  $\mathcal{T}$  for which

$$d(x, \Phi_A(x)) \leq r = |a - b|.$$

Let  $x_0$  be the fixed point of  $\Phi = \Phi_A$  on  $\mathcal{T}(M)$  and  $\tilde{x}_0$  its projection to  $\mathcal{T}$ . Note that the inequalities (6.42) imply that

$$B(x_0, \frac{r}{p+1}) \cap \text{Vert}(\mathcal{T}) \subset \text{Vert}(\mathcal{T})_{\nu, A} \subset B(x_0, \frac{r}{p-1}) \cap \text{Vert}(\mathcal{T})$$

where  $B(x_0, R)$  is the “ball”

$$(6.44) \quad d(x, x_0) \leq R$$

of radius  $R$  centered at the point  $x_0$ .

To refine this, we will consider several possible cases:

A)  $\tilde{x}_0$  is not a vertex in  $\mathcal{T}$ . Then  $\tilde{x}_0$  lies on a segment  $[\eta, \eta']$  with  $\eta, \eta'$  the closest vertices to  $\tilde{x}_0$ . Now consider the images  $\Phi(\eta), \Phi(\eta')$ . Since the geodesic  $[x_0, \Phi(\eta)]$  passes through the projection  $\tilde{x}_0$ , it also has to pass through either  $\eta$  or  $\eta'$  (but not both). Similarly for  $[x_0, \Phi(\eta')]$ . There are several subcases:

1)  $\eta \in [x_0, \Phi(\eta')], \eta' \in [x_0, \Phi(\eta)]$ . Apply  $\Phi$  to conclude that  $\Phi(\eta)$  lies in the geodesic from  $x_0$  to  $\Phi^2(\eta')$  and  $\Phi(\eta')$  in the geodesic from  $x_0$  to  $\Phi^2(\eta)$ . Note that if  $\Phi(\tilde{x}_0) \neq \tilde{x}_0$  and is, for example, between  $\tilde{x}_0$  and  $\eta'$ , then  $\Phi([x_0, \eta']) = [x_0, \Phi(\eta')]$  would pass first through  $\tilde{x}_0$ , then through  $\Phi(\tilde{x}_0)$ , and then through  $\eta$ . This contradicts the fact that this is a geodesic. A

similar contradiction is obtained if  $\Phi(\tilde{x}_0)$  is between  $\tilde{x}_0$  and  $\eta$ . We conclude that  $\Phi(\tilde{x}_0) = \tilde{x}_0$  and hence  $x_0 = \tilde{x}_0$ .

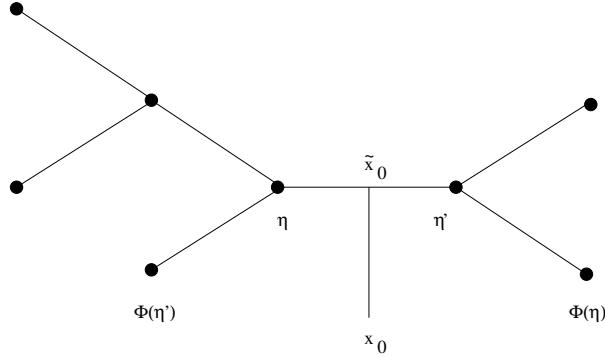


FIGURE 1. The case A1

By similar arguments, we deduce that the limit

$$\lim_{n \rightarrow \infty} [\Phi^{2n}(\eta), \Phi^{2n}(\eta')] = \lim_{n \rightarrow \infty} [\Phi^{2n+1}(\eta'), \Phi^{2n+1}(\eta)]$$

gives an apartment which is preserved (but flipped) by  $\Phi$ . Indeed,  $\Phi$  takes the half-apartment  $\lim_{n \rightarrow \infty} [x_0, \Phi^{2n}(\eta)]$  to  $\lim_{n \rightarrow \infty} [x_0, \Phi^{2n+1}(\eta)]$ .

Note that in this case, there are no half-apartments in the tree  $\mathcal{T}$  that are preserved by  $\Phi$ . Indeed, consider a vertex  $y$  in such a half-apartment and connect this to  $x_0$ ; the geodesic has to pass through either  $\eta$  or  $\eta'$ ; in either case, since the geodesic from  $x_0$  to  $\Phi(y)$  has to pass through the opposite point  $\eta'$ , resp.  $\eta$ , we obtain a contradiction. Recall that the set of half apartments in the tree can be naturally identified with the set of one-dimensional subspaces of the corresponding vector space  $\mathbb{F}((u))^2$ . Hence, we see that in case (A1) the  $\Phi$ -module given by the matrix  $A$  is simple.

Now suppose  $x$  is a vertex in  $\mathcal{T}$ . The geodesic  $[x_0, x]$  has to pass through either  $\eta$  or  $\eta'$ . Suppose that  $\eta' \in [x_0, x]$  (the other case is similar) and consider  $\Phi([x_0, x]) = [\Phi(x_0), \Phi(x)] = [x_0, \Phi(x)]$ . This contains  $\Phi(\eta')$  and therefore has to pass through  $\eta$  (since  $\eta \in [x_0, \Phi(\eta')]$ ). Therefore, the geodesic  $[x, \Phi(x)]$  passes through both  $\eta$  and  $\eta'$  (and also  $x_0$ ) and we have

$$\begin{aligned} d(x, \Phi(x)) &= d(x, x_0) + d(x_0, \Phi(x)) \\ &= (p+1)d(x, x_0). \end{aligned}$$

Hence, in this case,  $d(x, \Phi_A(x)) \leq r$  amounts to  $d(x, x_0) \leq r/(p+1)$  and we have

$$\text{Vert}(\mathcal{T})_{\nu, A} = B(x_0, \frac{r}{p+1}) \cap \text{Vert}(\mathcal{T}).$$

2)  $\eta \in [x_0, \Phi(\eta)]$ ,  $\eta' \in [x_0, \Phi(\eta')]$ . Then, we can see that the limits  $\lim_{n \rightarrow \infty} [x_0, \Phi^n(\eta)]$ ,  $\lim_{n \rightarrow \infty} [x_0, \Phi^n(\eta')]$  give two half-apartments that are both preserved by  $\Phi$ . As above, this implies that the  $\Phi$ -module given by  $A$  contains two 1-dimensional  $\Phi$ -submodules; we can easily see that these are distinct. Hence the  $\Phi$ -module given by  $A$  is decomposable.

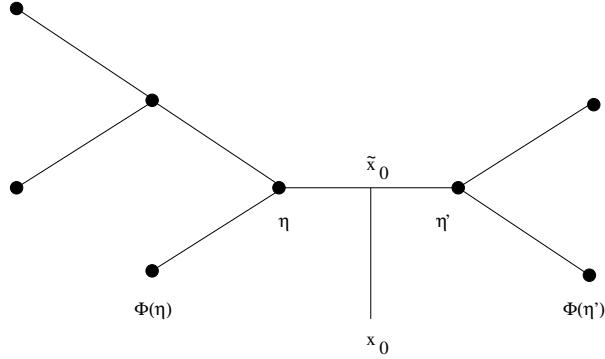


FIGURE 2. The case A2

Now suppose  $x$  is a vertex in  $\mathcal{T}$ . The geodesic  $[x_0, x]$  has to pass through either  $\eta$  or  $\eta'$ . Suppose that  $\eta \in [x_0, x]$  and in fact suppose that  $a \geq 0$  is the largest integer for which  $\Phi^a(\eta)$  is contained in  $[x_0, x]$ . Consider  $\Phi([x_0, x]) = [x_0, \Phi(x)]$  which has to contain  $\Phi^{a+1}(\eta)$ . Therefore, the geodesic  $[x, \Phi(x)]$  has to pass through  $\Phi^{a+1}(\eta)$ . We obtain

$$\begin{aligned} d(x, \Phi(x)) &= d(x, \Phi^{a+1}(\eta)) + d(\Phi^{a+1}(\eta), \Phi(x)) \\ &= d(x, \Phi^{a+1}(\eta)) + p \cdot d(\Phi^a(\eta), x). \end{aligned}$$

If  $x'$  is the projection of  $x$  to the half-apartment  $\lim_{\rightarrow n} [x_0, \Phi^n(\eta)]$ , then we can rewrite this distance as

$$\begin{aligned} d(x, \Phi(x)) &= d(\Phi(x), \Phi(x')) + d(\Phi(x'), \Phi^{a+1}(\eta)) + \\ &\quad + d(\Phi^{a+1}(\eta), \Phi^a(\eta)) - d(x', \Phi^a(\eta)) + d(x, x') \\ &= (p+1)d(x, x') + (p-1)d(x', \Phi^a(\eta)) + p^a d(\Phi(\eta), \eta). \end{aligned}$$

There is a similar expression if  $\eta'$  is in  $[x_0, x]$ . Hence,  $d(x, \Phi_A(x)) \leq r$  can be described as the union of two “thinning tubes” around the two half-apartments that are preserved by  $\Phi$ . Note that, in the above, when  $d(x, \Phi(x))$  is bounded, the possible values of  $a$  are bounded too.

3)  $\eta \in [x_0, \Phi(\eta)]$ ,  $\eta' \in [x_0, \Phi(\eta')]$  (the case  $\eta' \in [x_0, \Phi(\eta')]$ ,  $\eta \in [x_0, \Phi(\eta)]$  is symmetric). Then  $\lim_{\rightarrow n} [x_0, \Phi^n(\eta)]$  gives a half-apartment which is preserved by  $\Phi$ . As above, this implies that the  $\Phi$ -module given by  $A$  contains a 1-dimensional  $\Phi$ -submodule and, therefore, it is not simple. In this case, we can see that this is the unique half-apartment preserved by  $\Phi$ . Indeed, consider a vertex  $y$  in such a half-apartment  $\mathcal{A}'$  and connect this with a geodesic to  $x_0$ ; the geodesic has to pass through either  $\eta$  or  $\eta'$  and we can easily rule out  $\eta'$ . Now the geodesic  $[x_0, \Phi(y)]$  has to pass through  $\Phi(\eta)$ . Since  $\Phi(y)$  is also in  $\mathcal{A}'$ , we can conclude that  $\mathcal{A}'$  also contains  $\Phi(\eta)$ . Inductively,  $\mathcal{A}'$  contains  $\Phi^n(\eta)$  for all  $n$ . We conclude that the  $\Phi$ -module given by  $A$  is not simple and not decomposable.

Now suppose  $x$  is a vertex in  $\mathcal{T}$ . The geodesic  $[x_0, x]$  has to pass through either  $\eta$  or  $\eta'$ .

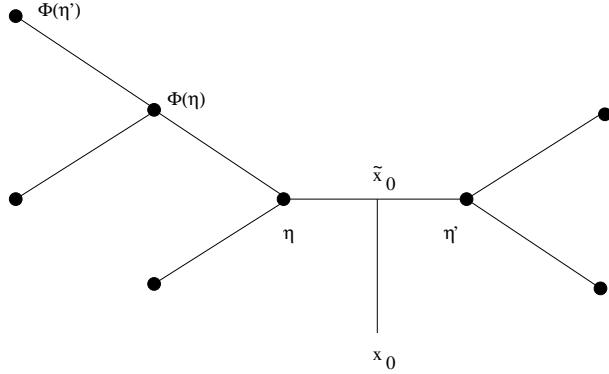


FIGURE 3. The case A3

- Suppose first that the geodesic  $[x_0, x]$  passes through  $\eta$ . In fact, suppose that  $a \geq 0$  is the largest integer for which  $\Phi^a(\eta)$  is contained in  $[x_0, x]$ . Then as before, we obtain

$$\begin{aligned} d(x, \Phi(x)) &= d(x, \Phi^{a+1}(\eta)) + pd(\Phi^a(\eta), x) \\ &= (p+1)d(x, x') + (p-1)d(x', \Phi^a(\eta)) + p^a d(\Phi(\eta), \eta) \end{aligned}$$

with  $x'$  the projection of  $x$  to the apartment  $\lim_{\rightarrow n}[x_0, \Phi^n(\eta)]$ . Hence again the set  $d(x, \Phi_A(x)) \leq r$  can be described for these vertices as a union of thinning tubes.

- Now suppose that the geodesic  $[x_0, x]$  passes through  $\eta'$ . Then an argument as in case (A1) gives

$$d(x, \Phi(x)) = (p+1)d(x, x_0) - 2d(\tilde{x}_0, x_0).$$

Hence for this kind of vertices this set  $d(x, \Phi_A(x)) \leq r$  is a ball around  $x_0$ .

B) Suppose now that  $\tilde{x}_0 = \eta$  is a vertex of  $\mathcal{T}$ . There are two subcases:

- 1)  $\tilde{x}_0 = \eta$  is not fixed by  $\Phi$ . Then,  $\lim_{\rightarrow n}[x_0, \Phi^n(\eta)]$  gives again a half-apartment that is preserved by  $\Phi$ . We can, in fact, see as before, that this is the unique such half-apartment. Hence, in this case, the  $\Phi$ -module given by  $A$  is not simple and not decomposable. Note that after replacing  $\mathbb{F}$  by a finite extension,  $x_0$  becomes of type B2 below.

If  $x$  is a vertex of  $\mathcal{T}$  then  $\eta \in [x_0, x]$ . If  $a \geq 0$  is the largest integer for which  $\Phi^a(\eta)$  is contained in  $[x_0, x]$  we obtain for  $d(x, \Phi(x))$  the same formula as in cases A2 or A3a. Hence in this case  $d(x, \Phi_A(x)) \leq r$  is a union of thinning tubes.

2)  $\tilde{x}_0 = \eta$  is fixed by  $\Phi$ , in other words the fixed point  $x_0$  is a vertex of  $\mathcal{T}$ . This corresponds to the homothety class of a lattice  $\mathfrak{M}_0$ ; we have  $\Phi(\phi^*\mathfrak{M}_0) = u^s\mathfrak{M}_0$  for some  $s$ . Denote by  $\{\eta_i\}_{i=0,\dots,p}$  its neighborhood vertices. The link of  $\eta$  is identified with the projective space of lines in  $\mathfrak{M}_0/u\mathfrak{M}_0$  and the action of  $\Phi$  on the link then corresponds to the action on the projective space given the linear action of  $u^{-s} \cdot \Phi$  on  $\mathfrak{M}_0/u\mathfrak{M}_0$ . Now observe that the geodesic  $[x_0, \Phi(\eta_i)]$  passes through  $\eta_i$  if and only if the action of  $\Phi$  on the link leaves the point of the link that is given by  $\eta_i$  fixed. Depending on whether the number of fixed points in the link is  $\geq 2$ , resp. 1, resp. 0, the  $\Phi$ -module is decomposable, resp. not simple and

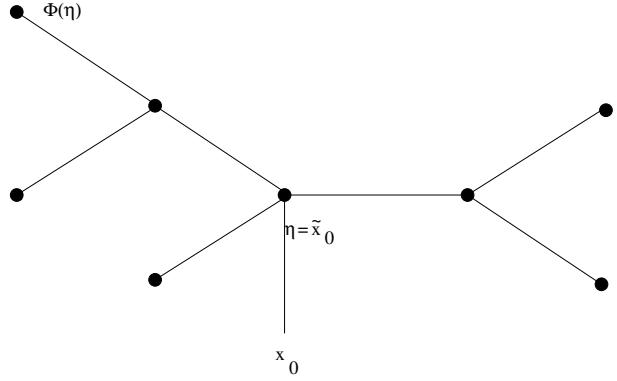


FIGURE 4. The case B1

not decomposable, resp. simple. Note that in the last case it is not absolutely simple, it disappears if  $\mathbb{F}$  is replaced by a finite extension.

Now let  $x$  be a vertex of  $\mathcal{T}$  and consider the geodesic  $[x_0, x]$  which has to pass through one of the vertices  $\eta_j$ . We distinguish cases according as  $\eta_j$  gives a fixed point of the link, or not.

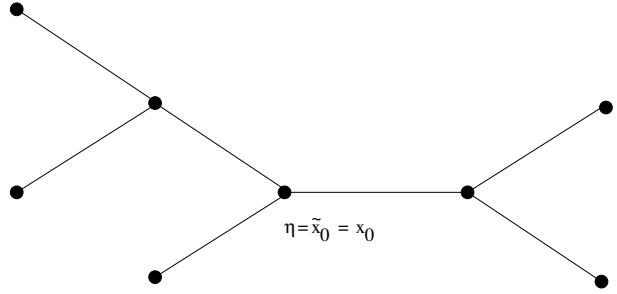


FIGURE 5. The case B2

- Suppose first that  $[x_0, x]$  passes through a vertex  $\eta_j$  with the corresponding point of the link fixed by  $\Phi$ . Then the argument of case A2 applies to obtain  $d(x, \Phi(x))$ . (It involves the largest integer  $a \geq 0$  such that  $\Phi^a(\eta_j)$  is in  $[x_0, x]$ .) For this kind of vertices  $d(x, \Phi_A(x)) \leq r$  is a union of thinning tubes.

- Now suppose that  $[x_0, x]$  passes through a vertex  $\eta_j$  with the corresponding point of the link not fixed by  $\Phi$ . Then the argument of case A1 applies to give

$$d(x, \Phi(x)) = (p+1)d(x, x_0).$$

For this kind of vertices  $d(x, \Phi_A(x)) \leq r$  is a ball around  $x_0$ .

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